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Chapter 1 Describing Empirical Knowledge

§ 1. Mathematics, Metaphysics, and the Natural World

Why is mathematics able to describe the phenomena of the natural world? Evidence that it can do so with a remarkable degree of success is all around us. Engineers use mathematics to design bridges, dams, roads, skyscrapers, computers, cell phones, aircraft, coffee makers, and, indeed, the greater part of all devices and structures we see and use every day. The laws of physics are expressed in mathematical equations. The other sciences also employ mathematics to one degree or another to describe and explain the phenomena they study (albeit few of the other sciences use mathematics to the degree and extent it is used in physics).

Yet mathematics is indisputably the product of human minds whereas the natural world is not. Furthermore, you can search the world over and you will never find a single sensible experience in which you have any direct physical encounter with any object of pure mathematics. A mathematical point, line, or circle is nowhere to be found; nor is a transcendental number, a mathematical hyperspace, or any other denizen of what we will call "the mathematical world." The only places where we find any immediate connection between the mathematical world and the world of physical nature are in the minds of human beings whose understandings of natural phenomena call upon supersensible mathematical objects to describe and explain them. By "supersensible object" I mean an object that can never be experienced by our senses or by any instrument capable of extending our senses (e.g., a microscope that allows us to see bacteria invisible to the naked eye or a telescope that allows us to view the rings of Saturn). It seems that mathematical objects exist outside of physical nature. Yet, if this is so, how is it possible for us to successfully understand the latter by means of the former? Is this not a fundamental paradox?

This treatise is written around the theme that, although many find a paradox here, it is not a *fundamental* paradox. It has a resolution, and this resolution is found by understanding more fully what mathematics is, what science endeavors to do with it, and what *practical* connections we make between the two. We will examine relationships between math and science and, by doing so, cultivate a practical understanding of how to *invent* mathematical objects, *bring them into* understanding nature, and *use* them to improve human lives and conditions.

There are some people who protest the notion that mathematical objects are *brought into* nature. They believe that mathematical objects are not mere human inventions; that they are somehow "real" and somehow "exist" in nature. There is both truth and falsity in these claims. People who make these claims are not-wrong, but they are also not-correct. The objects of mathematics *are* "real"; the question is: *in what way* are they real? The objects of mathematics *do* "exist in nature"; the question is: *in what way* do they exist? This treatise will answer both of these questions. Once you understand these answers, you will find yourself better able to use mathematics to understand the world around you if you wish to do so. You will also find yourself better able to recognize when a mathematical description is *not* a description of empirical nature but is instead merely an ungrounded speculation disconnected from physical reality.

Some people hold a mystic view of mathematics. This is usually called mathematical Platonism, named after the Greek philosopher Plato. Mathematicians Davis and Hersh wrote,

> Mathematical Platonism is the view that mathematics exists independently of human beings. It is "out there somewhere," floating around eternally in an all-pervasive world of Platonic ideas. Pi is in the sky. . . . The universe will have imposed essentially the same mathematics upon Galaxy X-9 as upon terrestrial men. It is universal. In this view, the job of the theorist is to listen to the universe sing and record its tune. [Davis & Hersh (1981), pp. 68-69]

This view is embraced by many professional mathematicians and, over the past several decades, by an alarmingly growing number of theoretical physicists and teachers. I use the word "alarming" here because this view is a serious metaphysical blunder that leads to the fantasies and errors of false knowledge. As

Will Rogers famously said, "It ain't what we don't know that gets us into trouble; it's what we know that ain't so." Mathematical Platonism does no harm to mathematics, but it can do great harm to the natural sciences.

The word "metaphysic" means nothing more and nothing less than "the way one looks at the world," and so subscribing to mathematical Platonism is nothing else than a subscription to a particular metaphysic. It is a metaphysic centered around objects – mathematical objects in this case – and for this reason it is called an "ontology-centered metaphysic." An ontology is a constituted system of all concepts and principles: (1) related to understanding objects in general; and (2) regarded as a science of the properties of all things in general. An ontology-centered metaphysic is a way of looking at the world that is grounded in ideas and principles of an ontology, in relationship to which epistemology[1] is derivative and grounded in objects.

With overwhelming likelihood, *your own* personal metaphysic through which you view the world is an ontology-centered metaphysic of one kind or another. Scientific studies of the development of intelligence in children demonstrate that every normally-developing human being views the world through a lens of naive realism in infancy [Piaget (1929)]. It is by looking at the world through this lens that human beings come to construct, each for himself, concepts and habits of thinking that produce a personal ontology-centered metaphysic [Wells (2009), chap. 1], [Wells (2012), chap. 3, pp. 53-58]. Piaget found,

> The child may be aware of the same contents of thought as ourselves but he locates them elsewhere. He situates in the world or in others what we seat within ourselves, and he situates in himself what we place in others. . . . The child is almost as well aware of [percepts, images, words, etc.] as we are but he gives them an entirely different setting. For us, an idea or a word is in the mind and the thing it represents is in the world of sense perception. Also words and certain ideas are in the mind of everybody, while other ideas are peculiar to one's own thought. For the child, thoughts, images and words, though distinguished to a certain degree from things, are nonetheless situated in the things. The continuous steps of this evolution may be assigned to four phases: (1) a phase of *absolute realism*, during which no attempt is made to distinguish the instruments of thought and where objects alone appear to exist; (2) a phase of *immediate realism*, during which the instruments of thought are distinguished from the things but are situated in the things; (3) a phase of *mediate realism*, during which the instruments of thought are still regarded as a kind of things and are situated both in the body and the surrounding air; and finally (4) a phase of *subjectivism* or *relativism*, during which the instruments of thought are situated within ourselves. In this sense, then, the child begins by confusing his self – or his thought – with the world, and then comes to distinguish the two terms one from another. [Piaget (1929), pp. 125-126]

This naive realism works well enough for young children within the limited scope of their experiences so that realist habits of thinking become deeply ingrained in their minds. These habits of thinking then go on to "give their own color" to everything the person perceives and experiences – and this is what a personal metaphysic does. Every one of us begins life as a naive realist and the great majority of all people never develop past this habituated ontology-centered realism.

There is a fatal shortcoming in every ontology-centered metaphysic. This shortcoming is that ultimately the most fundamental of its objects – those from which all other objects are constituted – are not only unknown but unknowable. They have to be treated as occult qualities and it is for this reason that every ontology-centered metaphysic eventually terminates in mysticism. Isaac Newton wrote,

> [The] Aristotelians gave the name of occult qualities, not to manifest qualities, but to such qualities only as they supposed to lie hid in bodies, and to be the unknown causes of manifest effects. . . . Such occult qualities put a stop to the improvement of natural philosophy, and

[1] the theory of knowledge, i.e. a science dealing with the sources, scope, and boundaries of human knowledge and reasoning.

> therefore of late years have been rejected. To tell us that every species of things is endowed with an occult specific quality by which it acts and produces manifest effects is to tell us nothing. [Newton (1704), pg. 401]

Like almost everyone else, Newton subscribed to an ontology-centered way of looking at the world. The mysticism that inevitably followed from this was rooted in his religion. He put it this way:

> All these things being considered, it seems probable to me that God in the Beginning formed Matter in solid, massy, hard, impenetrable, moveable particles, of such sizes and figures, and with such other properties, and in such proportion to space, as most conduced to the End for which he formed them; and that these primitive particles, being solids, are incomparably harder than any porous bodies compounded of them; even so very hard as to never wear or break into pieces, no ordinary power being able to divide what God himself made one in the first Creation. [*ibid*., pg. 400]

This is, of course, a speculation proposed without one shred of empirical evidence to support it. Newton himself was careful to avoid using his "primitive particles" in his theories and confined himself to theorizing about what above he called "manifest effects." As he had put it earlier,

> Hitherto we have explained the phenomena of the heavens and of our sea by the power of gravity, but have not yet assigned the cause of this power. . . . But hitherto I have not been able to discover the cause of those properties of gravity from phenomena and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. [Newton (1687), pp. 442-443]

Conjectures based upon ontology-centered presuppositions about objects are sometimes useful for solving more immediate problems or deducing descriptions of phenomena; but they are also misleading in subtle but crucial ways that later produce paradoxes in scientific reasonings. It is not incorrect to say that ontology-centered ways of looking at the world produce important errors that might not show up for a long time but which do, eventually, come to be discovered. When they do, the effects they have on scientific theories can be devastating. Physicist and Nobel Laureate Richard Feynman said:

> That reminds me of another point, that the philosophy or ideas around a theory may change enormously when there are very tiny changes in the theory. For instance, Newton's ideas about space and time agreed with experiments very well, but in order to get the correct motion of the orbit of Mercury, which was a tiny, tiny difference, the difference in the character of the theory needed was enormous. The reason is that Newton's laws were so simple and so perfect, and they produced definite results. In order to get something that would produce a slightly different result it had to be completely different. In stating a new law you cannot make imperfections on a perfect thing; you have to have another perfect thing. So the differences in philosophical ideas between Newton's and Einstein's theories of gravitation are enormous.
>
> What are these philosophies? They are really tricky ways to compute consequences quickly. A philosophy, which is sometimes called an understanding of the law, is simply the way that a person holds the law in his mind in order to guess quickly at the consequences. [Feynman (1965), pp. 168-169]

In other words, what Feynman called "a philosophy" is nothing else than a metaphysic. All of us make one for ourselves beginning in early childhood, and all of us think, reason, and understand in terms of the personal metaphysic each one of us constructs for himself.

The ultimate mysticism to which ontology-centered theories doom science might seem to be a severe indictment of science generally. Fortunately, it is not necessary to adopt an ontology-centered metaphysic

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in order to understand nature and the universe. The alternative is to adopt an epistemology-centered metaphysic in which understanding of objects is grounded in understanding the nature of human cognition based on constructs that are necessary for the possibility of experience as human beings come to know it. The great 18th century philosopher Kant was the first to understand the role that a metaphysic plays in our understanding of nature and to propose an epistemology-centered metaphysic for it. Kant wrote,

> I should think that the examples of mathematics and natural science, which have become what they are now through a revolution brought about all at once, were remarkable enough that we might reflect on the essential elements in the change in the ways of thinking that has been so advantageous to them and, at least as an experiment, imitate it insofar as their analogy with metaphysics, as rational knowledge, might permit. Up until now it has been assumed that all our knowledge must conform to the objects, but all attempts to find out something about them *a priori* through ideas that would extend our knowledge have, on this presupposition, come to nothing. Hence let us once try whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognitions, which would agree better with the requested possibility of an *a priori* knowledge of them, which is to establish something about objects before they are given to us. This would be just like the first thoughts of Copernicus, who, when he did not make good progress in the explanation of celestial motions if he assumed that the entire celestial host revolves around the observer, tried to see if he might not have greater success if he made the observer revolve and left the stars at rest. [Kant (1787) B: xvi]

This change in "the way one looks at the world" has come to be called "Kant's Copernican Revolution" and it gave birth to a new epistemology-centered system of metaphysics called "the Critical Philosophy."

This kind of metaphysic is ultimately practical – i.e., fundamental meanings are based on what one *does* with one's ideas and objects or *how* one comes to understand them. An idea that cannot be used in practice is properly called a *useless* idea. An epistemology-centered metaphysic works by reducing all *real* definitions to *practical* explanations. As this treatise explains, all our connections between objects of mathematics and the sciences of empirical nature are defined in this way. Not only is that so, but decades of painstaking research into the development of intelligence in children concludes that the meanings given to *all* objects by human beings are of this character:

> Meanings result from an attribution of assimilation schemes to objects, the properties of which are not "pure" observables but always involve an *interpretation* of the "data." In accordance with the classic definition of schemes ("a scheme is what can be repeated and generalized in an action"), we shall say that the meaning of an object is "what can be done" with the object . . . However, meanings are also what can be said of objects, i.e., descriptions, as well as what can be thought of them . . . As for actions themselves, their meaning is "what they lead to" according to the transformations they produce in the object or situation on which they bear. Whether predicates, objects, or actions are involved, meanings imply that the subject's activities interact either with an external physical reality, or with a reality the subject himself has previously generated, as in the case of logico-mathematical entities. [Piaget & Garcia (1987), pp. 159-160]

This finding would not have surprised Kant. He noted,

> What is an Object? [It is] that whose representation is an embodiment of several predicates belonging to it. . . . An Object is that in the representation of which various others can be thought as synthetically combined. [Kant (1776-95) 18: 676]

The immediate benefits of an epistemology-centered "way of looking at the world" are elimination of mysticism in science and a grounding of our understanding of scientific concepts in practical real-world terms.

The epistemology-centered metaphysic used in this treatise is called "the Critical Philosophy" and it is descended directly from Kant's pioneering work [Wells (2006)]. With it we shall be able to achieve a sound and practical understanding of what mathematics is and how it is used in science. You should, however, be forewarned that the greatest difficulty faced by people who strive to study the Critical Philosophy is the difficulty of breaking your own ontology-centered habits of thinking. Yet do this you must or you will be unable to correctly understand this epistemology-centered way of looking at the world.

§ 2. The Craft of Mathematics

As a first step toward understanding what mathematics is and how it is used in science, let us survey how mathematics developed out of practical human needs. What this survey shows us is that mathematics originally developed very slowly over a long period of time as or within *crafts* that satisfied practical purposes of early ornamentation, textile art, astronomy, commerce, architecture, and government. A **craft** is the practice of some special art. An **art** in general is the disposition or modification of things by human skill to answer an intended purpose. **Skill** is the ability to practice a craft. It took a very long time for "mathcraft" to develop to the point where people today would recognize its practices as anything like a "science" or be able to use its natal ideas to compose "theories."

If one were to insist that what we should identify by the name "mathematics" be recognizable in modern form, that restriction would leave many of its primal ideas and practices out of the picture altogether and would almost certainly obscure a large part of the meanings that gave purpose to mathematics and set the direction its development later followed. Smith remarked,

> Such a limitation, however, would not be a satisfactory one, for it would withdraw from our consideration those early steps in the development of the science which have great interest to the student and which are of value in considering the education of the individual. [Smith (1923), pg. 2]

As an example, consider the question, "What is a number?" Most of us first encounter the *word* "number" as very young children and are so habituated to using this term that we have long forgotten the practical contexts and connotations within and by which we learned about this idea. It seems to us now a trivial question unworthy of further inquiry. It might therefore come as a surprise, or even a bit of a shock, to find that the word "number" seems to have no *general* dictionary definition that can explain to us why, for example, "natural numbers" and "real numbers" are both "numbers." What we find instead of a definition are practical example *usages* in which the word "number" appears. The *Oxford Concise Dictionary of Mathematics* [Clapham (1996)] does not contain an entry for "number." *The Penguin Dictionary of Mathematics* [Nelson (2003)] contains entries for "natural number," "complex number," "cardinal number," and "ordinal number" but does not define what a "number" is or explain why all these different "species of numbers" are all "numbers." What is it that stands as genus to these species of "numbers"?

It isn't that mathematicians aren't concerned about "what a number is." In his classic textbook on advanced mathematics, renowned scholar Patrick Suppes wrote,

> The working mathematician, as well as the man in the street, is seldom concerned with the unusual question: What is a number? But the attempt to answer this question precisely has motivated much of the work by mathematicians and philosophers in the foundations of mathematics during the past hundred years. Characterizations of the integers, rational numbers and real numbers has been a central problem for the classical researches of Weierstrass, Dedekind, Frege, Peano, Russell, Whitehead, Brouwer, and others. [Suppes (1972), pg. 1]

Any question that has defied the ability of so many of the best minds in mathematics to satisfactorily answer deserves our respect. A definition that references itself is not a definition, but this is what we find in the many stated "definitions" of "number." As one example, Bertrand Russell wrote,

> *Number* is what is characteristic of numbers, as *man* is what is characteristic of men. [Russell (1919), pg.. 1]

This simply doesn't pass muster as a meaningful definition because it begs the question. This "definition" of "number" requires you to already know what "numbers" are.

Today you seldom hear anyone speak of "mathematical craftsmanship" or "the art of mathematics." There are some people who think calling mathematics a "craft" is demeaning to it, and there are others who point to the rigor of modern mathematical proofs and say this proves mathematics is an objective science not influenced by subjective factors such as those that influence "fine art." In this treatise I am going to make a case for regarding mathematics in terms of craftsmanship and for the fundamental role of subjective judgments in mathematics. But first let us take a look at present day views of mathematics.

The Oxford Dictionary tells us "mathematics" is "the abstract science of number, quantity, and space." A century ago this same source was describing it as "that abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations." More recently, Nelson tells us mathematics is "the study of numbers, shapes, and other entities by logical means" [Nelson (2003)]. Others claim "mathematics has no generally accepted definition." Schoolchildren take the pragmatic attitude that "mathematics" is whatever their mathematics teachers say it is. Few would argue "mathematics" is unimportant in people's daily lives but what *is* this thing nearly everyone regards as important?

Is mathematics a science? To answer this we must first understand what a "science" is. The Critical real explanation of a "science" is "a doctrine constituting a system in accordance with a principle of a disciplined whole of knowledge" [Kant (1786) 4: 467]. Although many today are unaware of it, Kant was the first to draw the distinction between "natural philosophy" and "science" we still use today. He further distinguished between the terms "historical doctrine of nature," "natural history," and "natural science":

> [The] doctrine of nature would be better arranged into *historical doctrine of nature*, which contains nothing but systematically ordered facts about natural things (and would in turn consist of *natural description*, as a classification system for them according to their similarities, and *natural history*, as a systematic presentation of them at various times and places), and natural science. Natural science would now be either *properly* or *improperly* so-called natural science, where the first treats its objects wholly according to *a priori* principles, the second according to laws of experience. [*ibid*., 4: 468]

Today we generally refer to Kant's "properly so-called natural science" as "rational science" carried out by means of hypotheses and theories, and his "improperly so-called natural science" as "empirical science" carried out by means of observations and experiments.

We usually use the word "doctrine" to denote something that is taught or to denote a body of principles in a branch of knowledge. These are somewhat vague ideas, however, and Kant had a crisper definition, namely, a doctrine is "that theoretical knowledge in which one comes across the grounds for how an object-matter can be trained up or the rules hit upon according to which a good product can be produced" [Kant (early 1770s) 24: 228]. This is an explanation that embraces both theoretical and practical knowledge.

Mathematics is certainly something that can be taught and so is a doctrine in that sense. What often goes unappreciated by most people is that the principal work of professional mathematicians involves producing "good products" – that is, producing new mathematical structures and objects that are useful not only to the mathematician but, sooner or later, to scientists, engineers, and others whose work involves the application of mathematics. Therefore it embraces the practical as well as the theoretical.

Consequently, there are two co-equal connotations of mathematics. In the *practical* connotation mathematics is a tool for systematic construction of practical knowledge. But, because the objects of pure mathematics are all supersensible objects we never experience directly and immediately, what is meant by "construction of knowledge" by means of mathematics? Nelson's "definition" given above refers to numbers, shapes, and "other entities." What "other entities"? Here the scope of things mathematics covers is vast and seems to be without definable limitation. Is there a general way to understand or state what it is that is being constructed adequate to take in this seemingly limitless scope? Kant tells us there is. From the theoretical connotation mathematics is *rational knowledge through the construction of concepts* [Kant (*c*. 1790-91) 28: 532]. If you consider the vast expanse of concepts people are capable of coming up with, the dictionary descriptions of mathematics we looked at earlier greatly underestimate the scope of things mathematics is capable of commanding.

However, mathematics, at least in recognizable form, is not a haphazard outcome. There are principles involved in its constructed concepts, and these principles bring to it a discipline that is applied to everything in it. Thus we find that mathematics *does* come under the Critical explanation of a science. As Kant put it,

> Mathematics is the science of the construction of concepts [Kant (1776-95) 18: 141].

Viewed in Kant's way, "mathematics" therefore has to be seen as an outcome of human activities. Let us take a look at some examples of these activities generally regarded today as propaedeutic to the invention of mathematics *as a science*.

§ 3. Measuring and Counting

We do not know when human beings first undertook activities that eventually came to be considered part of mathematics because this is lost in the darkness of prehistory. Two such activities are measuring and counting, both of which predated the invention of numbers [Smith (1923), pp. 6-8]. At its root, ***measuring*** means comparing a thing to another thing to determine something about the first in regard to some purposive standard defined by the second. Understood in this way, measuring is an ability not unique to human beings. For example, a lion looking at a buffalo cow and its calf is capable of deciding it prefers the calf to the cow as prey. This is an example of what we will call a *qualitative measurement*. It presumes too much to say the lion *knows* the calf is "smaller than" the cow or the cow is "bigger than" the calf. Observation of lion behaviors only suffices to tell us individual lions in the wild preferentially go after a calf instead of its mother. Whether this is instinctive behavior or learned behavior is irrelevant to our definition of "measuring." Because we observe this capacity in species that predate the appearance of anatomically modern humans, the general phenomenon of measuring is older than our species.

Measuring is a capacity for assessing or regulating. We see this reflected in language by sentences like, "With what measure you mete, it shall be measured to you again." We also see it in tools or instruments descended from prehistoric instruments such as tally sticks and knotted cords. Archeologists have found tally sticks dating as far back the Old Stone Age. The oldest of these artifacts so far discovered is the Lebombo bone found in South Africa, which has been radiocarbon dated to between 43,000 to 44,200 years ago [d'Errico *et al*. (2012)]. Keeping a record or account of something by means of tallying is a primitive "counting" or "sequencing" operation that does not require the idea of "numbers" (i.e., the so-called "natural" or "counting" numbers – 1, 2, 3, etc.) but does require ideas of composition, correspondence and conservation. One of the best known present day examples, regarded as a descendant of knotted cord tallying instruments, is the rosary bead (figure 1). Rosary beads are used by Catholics as a mnemonic aid in reciting a Scripture-based prayer called the Rosary. The Rosary requires a complex recitation that can involve as many as twenty "Mysteries of the Rosary" as well as strands at the beginning and end [Bail (2003)]. A rosary bead is a tool for keeping tactile track of where the devotee is in the prayer sequence so he or she can consciously meditate on the Mysteries during the recitation.

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Figure 1: Rosary beads.

Although its use is often described as "counting," in fact the rosary beads help the devotee sequence his or her way through the prayer without requiring any recourse to numbers. The devotee instead regulates his prayers through perceptual relationships that Kant called *judgments of perception*:

> *Empirical judgments, so far as they have objective validity*, are **judgments of experience**; those, however, that are *only subjectively valid* I call mere **judgments of perception**. The latter do not need a pure notion of understanding but only the logical connection of perceptions in a thinking subject. . . . All of our judgments are at first mere judgments of perception; they hold only for us, i.e., for our subject, and only afterwards do we give them a new reference, namely to an Object, and intend that the judgment should also be valid at all times for us and for everyone else [Kant (1783) 4: 298].

Objective reference of a bead to a place in the Rosary (e.g., one of "the ten hail Mary's") is an *ex post facto* judgment of the judgments of experience kind. This referencing is not necessary during an actual recital of the Rosary. The devotee's sensorimotor judgment of perception suffices in sequencing through the prayer. The regulating measurement by use the rosary beads is therefore a *qualitative* measurement.

The idea of measuring makes its connection with mathematics when the act of measuring produces what we will call a *quantitative measurement*. This requires a mathematical idea – namely that of a "quantity" – and that idea, in turn, requires the invention of *objective* counting and numbers. Although the idea of "counting" is perhaps the simplest and most basic idea found in mathematics, answering the question "what is counting?" turns out to be rather more complicated than you might think and involves some rather subtle concepts. One definition of "counting" is "the process of determining the number of elements of a finite set of objects." Few people raise any objection to this definition but, you should note, this definition requires the concept of a "number." This presents an issue, namely the "what is a number?" issue raised earlier.

Are we to suppose prehistoric humans first invented something called "numbers" and then discovered counting? Or are we to suppose they first invented "counting" and then invented "numbers"? Or did the ideas of "counting" and "numbers" arise together as a consequence of some common *process* of doing something? If the third answer is the correct one, the primitive real definition of both "counting" and

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"numbers" must be looked for within the context of that process. Evidence from both archeology (e.g., tally sticks) and psychology indicate that it is the third alternative above, namely that objective counting and numbers co-developed out of practical activities, which is the correct answer.

Almost certainly the first instruments used in the invention of objective counting and numbers were parts of the human body such as fingers, arms, and feet. Smith speculates, on the basis of anthropological evidence, that

> the primitive man could count only by pointing to the objects counted one by one. Here the object is all-important, as was the case with the early measures of all people. The habit is seen in the use of such units as the foot, ell, thumb, hand, span, barleycorn, and furlong. In due time such terms lose their primitive meaning and we think of them as abstract measures. In the same way the primitive words used in counting were at first tied to concrete groups, but after thousands of years they entered the abstract stage in which the group almost ceases to be a factor. . . . We say "seven," but we no longer think of a certain group of objects, nor do we demand such a group in order to count; we think of a word in an endless series, the word coming just after "six" and just before "eight." In the Malay and Aztec tongues, however, the number names mean literally one stone, two stones, three stones, and so on; . . . When a Zulu wishes to express the number six, he says "taking the thumb" (*tatisitupa*), meaning that he has counted all the fingers of the left hand and has begun with the thumb of the right hand. For seven he says "he pointed" (*u kombile*), meaning that he has reached the finger used in pointing. After the world abandoned the use of objects in counting, there developed the possibility of an infinity of number names, and hence in the course of time mankind was confronted by the necessity of classifying numbers and for naming them according to some simple plan. [Smith (1923), pp. 7-8]

Such examples show us quite clearly that early objective counting and numbering were squarely tied to simple sensorimotor activities, and in turn this demonstrates mathematics' origin as a craft activity. It also demonstrates that these activities were purposive because, e.g., Smith's Zulu would not "take the thumb" unless he had some reason to do so. By the time history arrives, mankind had already made the abstract advance from counting by fingers and toes to substituting defined standards for measuring and counting. Among the earliest of these is the cubit (the distance from the elbow to the tip of the longest finger, approximately equal to 18 inches or 1.5 feet). It is found in ancient Sumer (dating from *c*. 2900 BC) and in the 18th Dynasty of ancient Egypt (*c*. 2700 BC). The idea that one didn't necessarily have to use a body part to count or measure something, and that a rod of wood, stone, or metal could be used "to mean the same thing" (figure 2) was a great invention because it facilitated the development of commerce, trade, the invention of money and banking, architecture, and practical geometry.

Figure 2: The Nippur cubit bar from ancient Sumer *c*. 2650 BC.

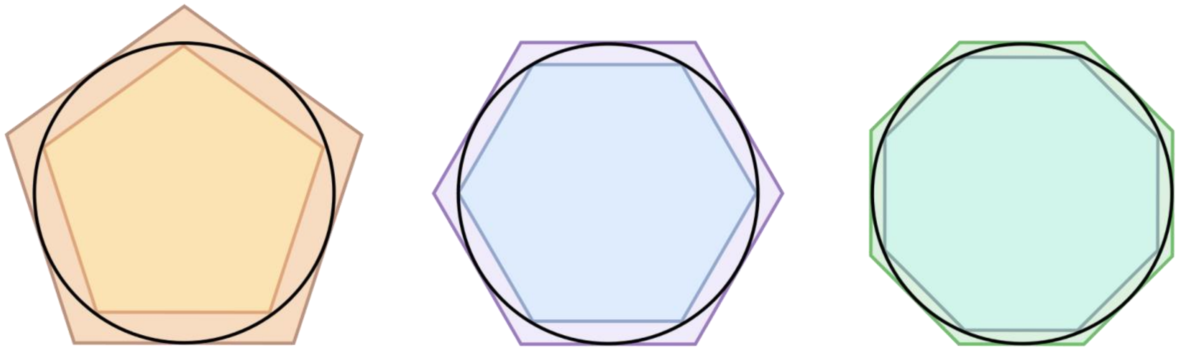

Figure 3: Illustration of the process used by Archimedes of Syracuse (*c*. 250 BC) to demonstrate that the ratio of the circumference of a circle to its diameter (pi) lies in the range between $223/71 < \pi < 22/7$. He did so by inscribing the interior and exterior of a circle with regular polygons and measuring the perimeters of these polygons. This method is sometimes called "the method of exhaustion" – basically because one keeps increasing the number of edges in the polygons used to inscribe the circle until he getting tired of continuing the algorithm. In Archimedes' case, he kept going until he was using 96-sided polygons. Isaac Newton used basically the same idea in his invention of calculus, which he called "the method of first and last ratios of quantities" [Newton (1687), pp. 31-39]

Millennia later this great abstract leap made possible the invention of the so-called "real numbers" and the discovery that so-called "irrational numbers" which cannot be expressed as the ratio of two whole numbers existed. (Irrational numbers are so called because they cannot be expressed as ratios). The invention of "rational numbers" (numbers defined to be ratios of whole numbers) did, of course, require the previous invention of the idea of "fractions." Stone tablets dated to *c*. 2400 BC tell us that fractions had already been invented by the time of the third dynasty of Ur in ancient Babylonia and were being used to compute interest on loans [Smith (1923), pg. 38]. Evidence like this further underscores the practical nature of mathcraft before its transition into the more familiar science we know today was begun by the ancient Greeks.

Psychology studies of the development of the idea of "numbers" in young children likewise demonstrates that very young children (ages 3 years and up) come to the concepts of "counting" and "numbers" by a gradual process in which compositions, correspondences, and conservations fuse into a union little by little [Piaget (1941)]. Piaget found he could distinguish the child's development of the idea of "numbers" into three distinguishable stages. In the first, the child does not understand number names in terms of quantification but, instead, regards "quantity" in terms of perceptual relationships. For instance, if you pour water from a wide short glass into a narrow tall glass, a stage 1 child will tell you there is "more" water in the second glass than there was in the first [*ibid*., pp. 5-13]. It is not until the third stage, somewhere in the range of 5 to 7 years, that the child is able to use the concept of numbers to quantify and establish correspondences between numbers and objects the way an adult does.

The mere fact that a very young child might be able to recite, "One, two, three, four," does not imply the child understands such ideas as "four is more than three," although he might tell you "four comes after three." Piaget found that in very young children,

> Thus not only the one-to-one correspondence, but also the actual enumeration, seem to the child at the first stage to be a less sure means of quantification than direct evaluation through global perceptual relationships (gross quantities). When he is required by his social environment to count up to a certain number, his counting is still only verbal and has no operational value. As for the one-to-one correspondence, it would be a great mistake . . . to consider it as being already a quantifying operation; it is as yet only a qualitative comparison. [Piaget (1941), pg. 25]

In other words, the mere ability to place things in a one-to-one correspondence is *not* sufficient to establish in the mind of a very young child the *mathematical* idea of a "number." An adult knows "one, two, three" and "do, re, me" both contain "3" words; to a four-year-old this is not a self-evident truth.

Likewise, the existence of Paleolithic tally sticks does not necessarily imply Paleolithic Man had already developed the idea of counting in contexts by which mathematics regards this idea. His tally stick might very well have been little or nothing more than a *tool* for making *practical* qualitative comparisons. This is not yet the same thing as the development of *theoretical* ideas of mathematics. Piaget wrote,

> The question to be considered is whether the development of the notion of conservation of quantity is not one and the same as the development of the notion of quantity. The child does not first acquire the notion of quantity and then attribute constancy to it; he discovers true quantification only when he is capable of constructing wholes that are preserved. At the level of the first stage, quantity is therefore no more than the asymmetrical relations between qualities, i.e., comparison of the type 'more' or 'less' contained in judgments such as 'it's higher', 'not so wide', etc. These relations depend on perceptions, and are not yet relations in the true sense, since they cannot be coordinated one with another in additive or multiplicative operations. This coordination begins at the second stage and results in the notion of 'intensive' quantity, i.e., without units, but susceptible of logical coherence. As soon as this intensive quantification exists, the child can grasp, before any other measurement, the proportionality of differences and therefore the notion of extensive quantity. This discovery, which alone makes possible the development of numbers, thus results from the child's progress in logic during these stages. [*ibid*., pg. 5]

By "the child's progress in logic" Piaget means such things as the ability to form relationships of class inclusion (e.g., "ducks are birds; sparrows are birds; therefore ducks and sparrows are birds"). Very young children do not understand logical implications of class inclusion. Among other things, this means that a very young child does not understand such "basic" mathematical propositions such as the distributive property of addition,

$$A + B + C = (A + B) + C = A + (B + C).$$

As an example,

> But while the child can carry out classifications of this sort, he is not able to understand the relationships of class inclusion. It is in this sense that his classifying ability is still preoperational. He may be able to compare subclasses among themselves quantitatively, but he cannot deduce, for instance, that the total class must necessarily be as big as, or bigger than, one of its constituent subclasses. A child of this age will agree that all ducks are birds and that not all birds are ducks. But then, if he is asked whether out in the woods there are more birds or more ducks, he will say, "I don't know; I've never counted them." [Piaget (1970), pp. 27-28]

Suppose we let A denote "birds" as a class. Therefore, "ducks are birds" is a statement that can be symbolized as "ducks = A." Similarly, "sparrows are birds" is also symbolized by "sparrows = A," and "robins are birds" is symbolized by "robins = A." Once ducks, sparrows, and robins are all grouped into the class "birds," to the child,

> Birds plus more birds equal birds. This means that distributivity does not hold within this structure. If we write $A + A – A$, where we put the parentheses makes a difference in the result. $(A + A) – A = 0$, whereas $A + (A – A) = A$. [*ibid*., pg. 28]

To fill in the steps in the child's thinking that Piaget omits in the above, if the child thinks $A + A – A = (A + A) – A = A – A$ then $A + A – A = 0$; but if he thinks $A + A – A = A + (A – A) = A + 0$ then $A + A – A = A$. His conclusion, in other words, is dependent upon how he "groups" or "classifies" *the objects* he is thinking about. "Understanding numbers" *includes* understanding an assortment of logical operations *by which numbers are mentally tied to objects*. It is not without reason that "the art of calculating" was called ***logistic*** by the ancient Greeks.

Psychology research of this kind tells us there is a fundamental problem with the definition of "number" given by Russell and Whitehead, i.e., "the number of a class is the class of all those classes that are similar to it" and "a number is anything which is the number of some class" [Russell (1919), pp. 18-19]. One of the consequences of this problem is displayed by the famous Russell Paradox, an example of which goes like this: "In a certain village there is one barber. The barber shaves everyone who does not shave himself, and he does not shave anyone who does shave himself. Who shaves the barber?" If we say, "the barber shaves himself" this contradicts the second premise. But if we say "someone else shaves the barber" this contradicts the first premise. Therefore, both possible answers are false – a paradox brought about because of way the villagers are *classified* into a set of "people who shave themselves" and "people who do not shave themselves."

Considerations such as this bring out an important difference between qualitative *measuring* (comparison of attributes or qualities of objects) and *numbering*. Piaget put it this way:

> How then are classes to be transformed into numbers? . . . As we saw in Chapters III-IV, Russell's solution to this problem is too simple. For him and his followers, two classes have the same number when there is a one-to-one correspondence between their elements. . . . The qualitative correspondence between the two classes F and F_1 merely means that these two classes have the same hierarchical structure, the same classification, but not the same number. As we saw in Chapters III-IV, there are various kinds of qualitative correspondence, which depend on the spatial position of the elements and have no numerical significance. . . . [If] we assert that any element in F can correspond to any element in F_1, we have the right to conclude that F and F_1 correspond numerically term for term, and that this correspondence defines the number . . . This number is not a 'class of classes', but the result of a new operation brought in from outside, which is not contained in the logic of classes as such. In fact, this 'quantifying' correspondence is only achieved by disregarding all the attributes in question, i.e., by disregarding the classes.
>
> In order to transform classes F and F_1 into numbers, the first essential condition is that their terms shall be regarded as equivalent from all points of view simultaneously. This is however contrary to our earlier statement with respect to classes. This brings us to the second condition: the equivalent terms must remain distinct. . . . That is to say that in addition to the inclusion $A + A' = B$, characteristic of classes, a principle of seriation is involved, and . . . seriation is merely an addition of differences, as distinct from addition of classes, which is an addition of elements that are equivalent from one given point of view.
>
> These two conditions are necessary and sufficient to give rise to number. Number is at the same time a class and an asymmetrical relation, the units of which it is composed being simultaneously added because they are equivalent and seriated because they are different from one another. In qualitative logic, the operational fusion of these two characteristics is impossible . . . Number, on the contrary, is the outcome of generalization of equivalence and generalization of seriation. [Piaget (1941), pp. 182-184]

Most people most of the time presume that when you "measure" something you automatically assign a "number" to it. The underlying psychology of human mental nature tells us this is not actually true, and that there is a complex combination of sensational judgments (judgments of perception) and object judgments (judgments of experience) participating in your thinking by which merely qualitative measurement is transformed into quantitative measurement – i.e., the assignment of a supersensible object we call "a number." The putting-together of the perceptual pieces defining the measured Object is, in Piaget's terminology, an equivalence scheme; but *counting up* these pieces to arrive at "a number" is a scheme of seriation. By "scheme" Piaget means "that which can be repeated and generalized in an act or action." Because a scheme is an action, this means "numbers" are given their primitive meanings through

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practical sensorimotor *operations*. Number-objects are defined by *actions*, not by other objects, are so are practical objects at their roots. The ancient Greek philosopher Heraclitus might have put it this way: "From the strain of binding opposites comes harmony" [Haxton (2001), pg. 31].

§ 4. Geometry and Its Abstract Objects

Geometry is the branch of mathematics concerned with describing the properties of figures in space. Like the rest of mathematics, its roots go back into prehistory and appear in geometric ornamentations in various arts. Smith tells us,

> A further prehistoric stage of mathematical development is seen in the use of such simple geometric forms as were suggested by the plaiting of rushes, the first step in the textile art. From there developed those forms used in clothing, tent cloths, rugs, and drapery which are usually found among primitive people. . . . Such decorations are not confined to the textiles of the people; they are equally prominent in architecture in all parts of the world. . . . The early pottery of Egypt and Cyprus shows very clearly the progressive stages of geometric ornament, from rude figures involving parallels to more carefully drawn figures in which geometric design plays a more important part . . . Art was preparing the way for geometry. [Smith (1923), pp. 15-16]

Figure 4 provides an ornamentation example from ancient Egypt before the time of the Pharaohs. It is not surprising or strange that geometric ornamentation found its way into ancient art. Psychologists use the term "gestalt" to refer to unified wholes, complete structures, configurations, shapes, and forms with which visual images are endowed by human perception. To us nature seems rife with lines, swirling patterns, and rounded shapes. Consider as an example the photograph of the night sky provided by figure 5 with its striking appearances of mountain ridgelines, mountain slope lines, dots of stars, and cloudy swirls of the Milky Way where we know no continuous solid lines actually exist. People are capable of imagining they can see objects such as the face of a "man in the moon" (figure 6) even though we know these objects are imaginary.

Before geometry became a formal part of the science of mathematics it became a practical craft used in surveying and architecture. The ancient Greeks transformed it from this into the mathematical science we

Figure 4: A predynastic Egyptian vessel from the period 4000 to 3000 BC.

Figure 5: Photograph of the night sky and the Milky Way.

Figure 6: Photograph of the full moon. Some people can imagine they can see the face of a man in this image.

know today. Isaac Newton provided a concise summation of the evolution of practical mechanics into geometry:

> Since the ancients . . . made great account of the science of mechanics in the investigation of natural things; and the moderns, laying aside substantial forms and occult qualities, have endeavored to subject the phenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics so far as it regards philosophy. The ancients considered mechanics in a twofold respect: as rational, which proceeds accurately by demonstration; and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. But as artificers do not work with perfect accuracy, it comes to pass that mechanics is so distinguished from geometry that what is perfectly accurate is called geometrical, what is less so is called mechanical. But the errors are not in the art but in the artificers. . . . [For] the description of right

> lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines but requires them to be drawn; for it requires that the learner should first be taught to describe them accurately before he enters into geometry; then it shows how by these operations problems may be solved. . . . Therefore geometry is founded in mechanical practice and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring. [Newton (1687), pg. 3]

The earliest known evidence of geometry dates back to around 3000 BC in the Indus Valley and in ancient Babylonia. Artifacts from this period include collections of empirically discovered principles of lengths, angles, areas, and volumes that apparently satisfied practical needs in surveying, architecture, construction, and other crafts. Other archeological evidence shows that it was known in ancient Egypt before *c*. 1850 BC. This evidence includes the Moscow Mathematical Papyrus (*c*. 1800 BC), the Egyptian Mathematical Leather Roll (17th century BC), and the Rhind Mathematical Papyrus (*c*. 1550 BC). These documents contain solutions to various mathematical problems that presumably were of practical interest to the ancient Egyptians. In addition to problems in geometry, they also present solutions to problems involving fractions and a version of the Pythagorean Theorem. They do not, on the other hand, present any clues as to *how* the Egyptians arrived at these solutions. I tend to think of them as being an ancient equivalent to "reference books" from which the Egyptian craftsmen and architects could look up the answers to frequently recurring problems. Newton would probably say their material pertained to "mechanics" rather than "geometry" as we today have come to regard the latter.

The Greeks learned this "practical art" from the Egyptians sometime in the 7th to 6th century BC. The most pronounced difference between Greek mathematics and that of other people was the introduction of organized and structured abstract principles about mathematics itself. The Egyptians, Phoenicians, and others clearly held their interests in mathematics subordinate to its practical applications. The Greeks introduced the study of mathematics *for the sake of* mathematics, and by doing so turned the craft into a proper science. So it is that Smith wrote,

> If mathematics means that "abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations," as the Oxford Dictionary defines it, then the history of mathematics cannot, strictly speaking, go back much further than the time of Thales (*c*. 600 BC) [Smith (1923), pp. 1-2].

The Greeks' revolutionary contribution to mathematics was their explicit introduction of idealized supersensible objects as the basis of theoretical mathematics [Thomas (1939)]. Included among these objects are mathematical points, lines, surfaces, angles, boundaries, and figures. These abstract objects, and others developed since then, dominated the science of mathematics ever after. From the time of Euclid's *Elements* (*c*. 300 BC) until the mid to late 19th century, mathematicians and other scientists regarded geometry as *the* basis of all mathematics[2,3]. The Greek root of the word "mathematics" is μαθημα ("*mathema*"), a word derived from a Greek word that means "that which is learned; lesson." In particular, μαθημα is what is learned *about* these ideal and supersensible objects. Moreover, the Greeks did not regard these objects as imaginary or in any way fictional; instead they made these objects bases of ontology-centered metaphysics and natural science. Around 100 AD, Nicomachus wrote,

---

[2] This view changed in the 19th century only after mathematicians were rocked by the discovery of non-Euclidean geometries and Riemann's discovery that there weren't just one or two non-Euclidean geometries but, indeed, that an unlimited number of them were "mathematically possible."

[3] After the fall of geometry as the accepted "foundation" of mathematics, mathematicians turned to set theory for its replacement. The Russell Paradox and other problems culminating in Gödel's incompleteness theorems [Gödel (1931)] made this effort unsuccessful too [Davis & Hersh (1981), pp. 330-338]. All these failures are outcomes of trying to use an ontology-centered metaphysic of one kind or another to set the foundations of mathematics.

> The ancients, who under the leadership of Pythagoras[4] first made science systematic, defined philosophy as the love of wisdom. Indeed, the name itself means this, and before Pythagoras all who had knowledge were called "wise" indiscriminately – a carpenter, for example, a cobbler, a helmsman, and in a word anyone who was versed in any art or handicraft. Pythagoras, however, restricting the title so as to apply to the knowledge and comprehension of reality, and calling the truth in this the only wisdom, naturally designated the desire and pursuit of this knowledge "philosophy," as being the desire for wisdom.
>
> He is more worthy of credence than those who have given other definitions, since he makes clear the sense of the term and the thing defined. This "wisdom" he defined as the knowledge, or science, of the truth in real things, conceiving "science" to be a steadfast and firm apprehension of the underlying substance, and "real things" to be those which continue uniformly and the same in the universe and never depart even briefly from their existence [Nicomachus (*c*. 100 AD), pg. 811].

If, as the Greeks believed, the supersensible objects of mathematics were somehow the "real things" of which the universe was composed, then indeed "the job of the philosopher was to listen to the universe sing and record its tune." It was this belief that mathematics was "the truth in real things" that injected an element of mysticism into the roots of mathematics and science. Fortunately for the welfare of humankind, both mathematics and science can and do accomplish a great many important things without relying upon mysticism. And, unfortunately, science from time to time does embark on flights of fancy precisely because of this mysticism. It is important to be able to recognize what is myth in science speculations.

Today this metaphysical underpinning is rarely formally taught in mathematics courses. Nonetheless it lingers on in the way many people tend to think about mathematics and its objects. It is for this reason important to take a look at how "points", "lines", etc. came to be defined.

Nelson's dictionary of mathematics defines a "point" as

> An element of geometry having position but no magnitude. A point in three dimensional space is defined by its coordinates (x, y, z). [Nelson (2003), "point"]

This certainly seems clear and simple enough at first glance when we think of it in terms of it being just a location in space and make "point" a synonym for "location." Provided one has a measurement procedure for determining "coordinate locations" it certainly can be "defined" by coordinates just as Nelson says. Metaphysical challenges arise only when one asks "what sort of ontological *object* is it?" Nelson's definition tells us a "point" is an "element" of geometry, and an "element," Nelson tells us, is

> a member of a set or group. [*ibid*., "element," def. 2]

But a "member" is

> Any of the individual entities belonging to a set [*ibid*., "member (element)"]

and a "set" is

> A collection of any kind of objects. [*ibid*., "set (class)"]

Once we put this string of dictionary definitions together, we are being told to regard a "point" as an "entity" of some kind. As an *individual* entity, it is singular. Putting all this together we find nothing in these definitions that differs from the definition laid down by Euclid in his *Elements*:

[4] Pythagoras lived *c*. 570 BC to *c*. 490 BC.

> A point is that which has no parts. [Thomas (1939), vol. I, pg. 437]

All this descriptive hairsplitting makes no practical difference as long as we do not care to pose any sort of relationship between an object of mathematics and any empirical object of nature. However, if we do wish to posit such a relationship then the ontology by which we regard a "point" can make a very big difference. For example, physicist and Nobel Laureate Richard Feynman said,

> On the other hand, I believe that the theory that space is continuous is wrong, because we get these infinities and other difficulties, and we are left with questions on what determines the size of all the particles. I rather suspect that the simple ideas of geometry, extended down into infinitely small space, are wrong. Here, of course, I am only making a hole and not telling you what to substitute. If I did, I should finish this lecture with a new law. [Feynman (1965), pp. 166-167]

If Feynman's conjecture that "space" is really "discontinuous" were to be accepted as correct, the impact on the science of physics would be wholesale. If would mean differential equations could no longer be used to describe the laws of physics and would have to be replaced by difference equations. It would raise fundamental problems in describing or offering to explain how "motion" of an object from one place to another is possible. It would require a rethinking of how to explain the composition of gross bodies and a host of other questions physics today regards as settled (more or less). In science, ontology makes a great deal of difference.

How do most people envision a "point" ontologically? Possibly the most typical way is to imagine some physical object – let us say, for example, a baseball – and then imagine this object becoming smaller and smaller and smaller until it becomes "smaller than anything" and "if it got any smaller it wouldn't exist at all." Thinking about a "point" this way is obviously a process you can carry out in your own mind and seems to be a more or less satisfactory way of envisioning "what a point would look like" if we could actually see one. It provides an "entity" object to go with the mathematical descriptions above.

But there are logical consequences accompanying this way of "looking at a point." A real baseball is made up of a number of constituent parts: a cork center; yarn windings; a cowhide cover; stitches (figure 7). If we use a baseball as a "model" for imagining a "point" *all of these constituent parts have to "disappear"* when the imaginary object "gets smaller than anything." If they didn't, the object would not

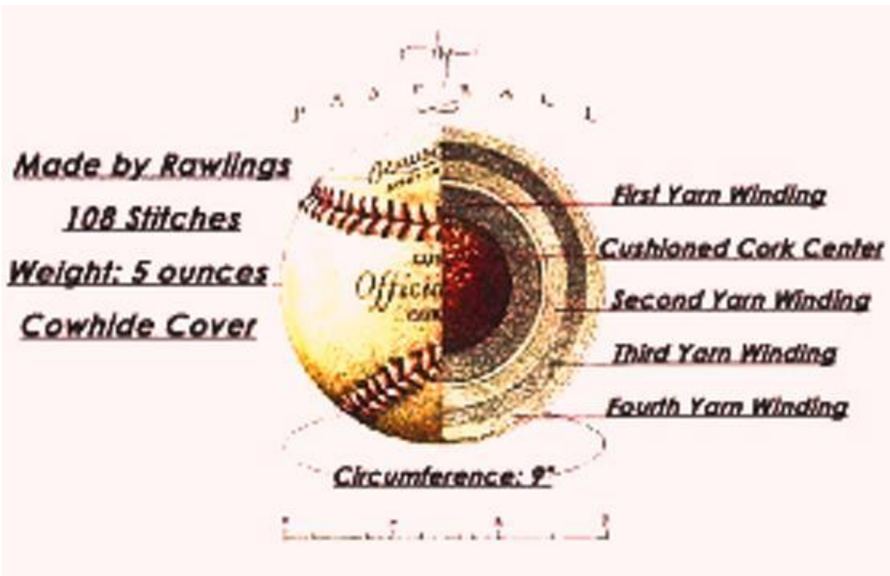

Figure 7: How a baseball is made.

"be smaller than anything" and we wouldn't "have reached the end" of our imaginary process of shrinking the baseball. The same logical consequence also applies to any other kind of model we might choose. A point is "that which has no parts." Once we say we have a "point baseball" we can no longer say anything *with objective validity* about what it is made of or how it is made up. What we are left with is no longer an *ontological* thing (object of physical nature) but, rather, an *epistemological* thing (object of knowledge).

Lest you think this argument is nothing but philosophical jibber-jabber, I will point out that the ontological issues raised by it are present in ongoing and very vexing questions about electrons. That electrons really do exist is accepted by all physicists. That they have a number of empirically determinable properties (charge, mass, etc.) is also universally accepted by physicists. These properties are described in Leonard & Martin (1980), pp. 2-48, as well as in various other physics textbooks. Leonard & Martin comment,

> Winston Churchill might have been talking about an electron rather than Russia when he termed it "a riddle wrapped in a mystery inside an enigma." It must be accepted at the outset that no one knows what an electron is. [Leonard & Martin (1980), pg. 2]

You don't hear or read very much about the vexing "problem of the electron" because no one has yet put forth any solution for it that the physics community generally accepts. But the problem (or problems) are there nonetheless. Physicists find ways of "working around" these problems. Feynman (along with Schwinger and Tomonaga) won his Nobel Prize for inventing such a workaround, the "theory of quantum electrodynamics" or QED. (I can suggest looking at this "workaround" through the lens of an epistemological branch of mathematics I introduce later that goes by the name "set membership theory").

The same general epistemological consideration arises in other geometric ideas. Euclid tells us

> A line is a length without breadth.
> The extremities of a line are points.
> A surface is that which has length and breadth only.
> The extremities of a surface are lines.
> A boundary is that which is the extremity of anything. [Thomas (1939), vol. I, pp. 437-439]

For example, we can model a "line" in our imagination by taking a stick and imagining it becoming "skinnier and skinnier until, if it became any skinnier, it wouldn't exist at all." We can model a surface by imagining a table top and then imagining it becoming "thinner and thinner until if it became any thinner it wouldn't exist at all."

These and all other objects of geometry share the common feature that they are idealized objects of one's imagination and are never physically encountered in anyone's actual experience. As such, they have *epistemological* significance (i.e., they are objects of knowledge) but do not have any *ontological* significance (i.e., they are not actual objects existing in physical nature). We do not know for certain who first developed this thorough-going abstraction for the concepts of Egyptian practical geometry but it is widely speculated to have been Thales of Miletus (*c*. 624 BC – *c*. 546 BC), one of the Seven Sages of Ancient Greece and the first well-known Greek philosopher, mathematician, and astronomer. Smith wrote of him,

> Up to this time geometry had been confined almost exclusively to the measurement of surfaces and solids, and the great contribution of Thales lay in suggesting a geometry of lines and in making the subject abstract. With him we first meet with the idea of a logical proof as applied to geometry, and it is for this reason that he is looked upon, and properly so, as one of the great founders of mathematical science. . . . Without Thales there would not have been a Pythagoras . . . and without Pythagoras there would not have been a Plato [Smith (1923), pg. 68].

If Smith is correct then before Thales geometry was a practical art that had not yet attained the status of a

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science. Its Greek word, γεομετρια (*geometria*), derives from a Greek phrase that means "to measure the earth."

5. Mathematics and Measurement

If "arithmetic" means "computation using numbers and simple operations such as addition, subtraction, multiplication, and division" (as Plato said [Plato (date uncertain), Bk. VII, 522C], [Nicomachus (*c*. 100 AD), pg. 813]) then arithmetic and geometry have both been tied to measurements and counting since their earliest appearances in the archeological record. In the case of arithmetic this is obvious when we regard counting as an arithmetic method of measuring (e.g., four goats; four fingers; 4). That this is so in the case of geometry is obvious from the meaning of word γεομετρια.

There is ample reason to suppose arithmetic was invented before geometry was. After all, the most primitive form of surveying – e.g. setting out an area ten paces wide by twenty paces long – is a counting operation. At the same time, there is little reason to think the ancients occupied themselves with questions like "is geometry derived from arithmetic?" Rather, they seemed to have used one or the other (arithmetic or geometry) for solving similar sorts of problems according to the preference of the problem solver. For example, we know from the Berlin Papyrus 6619 (written between 2000 and 1786 BC) that the ancient Egyptians knew enough about what later came to be called the Pythagorean Theorem to solve at least some problems without resorting to the use of triangles. It contains an arithmetic solution to a problem stated as, "the area of a square of 100 is equal to that of two smaller squares. The side of one is ½ plus ¼ the side of the other." The particular problem solved had two smaller squares, one of them 8×8 and the other 6×6, and the solution provided is given in the form of a single equation. It is a "geometry problem" but geometry is not used as the solution method[5]. The oldest existing *geometric* proof of the Pythagorean Theorem is found in Euclid's *Elements* (*c*. 300 BC) [Thomas (1939), pp. 178-185]. This proof uses construction of geometric figures (figure 8) to prove the square of the length of the hypotenuse of a right triangle (c^2) is equal to the sum of the squares of the other two sides ($a^2 + b^2$).

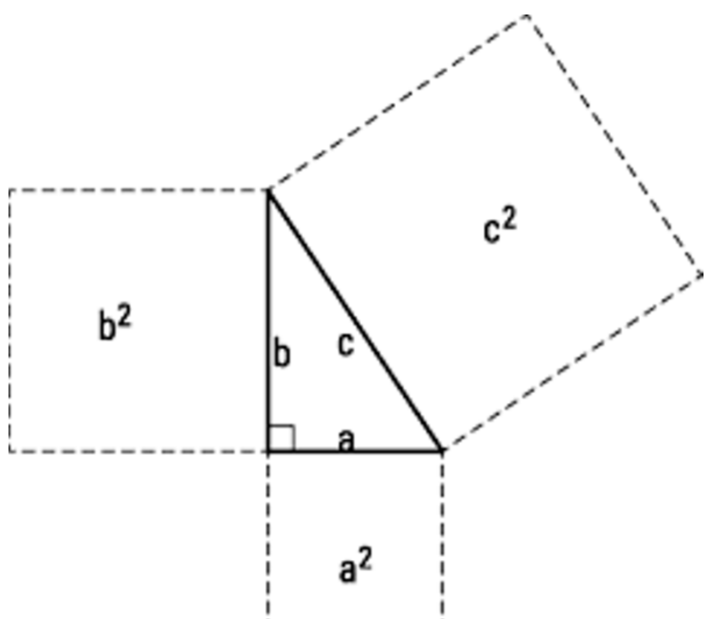

Figure 8: Diagram used by Euclid to prove the Pythagorean Theorem. The diagram is called "squaring a triangle."

[5] Today we would call their solution an "algebraic solution," but "algebra" as a generalization of the idea of "arithmetic" was first invented by Arab scholars around 830 AD.

At the beginning of this chapter I raised the question "Why is mathematics able to describe the phenomena of the natural world?" The first clue to answering this question is found in this common trait that our knowledge of arithmetic and geometry both fundamentally derive from making measurements and that, without inventing ways to make measurements, even the simplest and seemingly most obvious axioms of mathematics would not "be self evident truths." Consider Euclid's five "common notions":

> Things which are equal to the same thing are equal one to another;
> If equals are added to equals, the wholes are equal;
> If equals are subtracted from equals, the remainders are equal;
> Things which coincide with one another are equal to one another;
> The whole is greater than the part. [Thomas (1939), pg. 445]

Without the ability to make measurements the very idea of "things which are equal" is meaningless and therefore not one of these five "common notions" could mean anything at all. Implicit in all of them is the idea of making some kind of measurement (even if only by counting). Perhaps this strikes you as a trivial matter but, if so, bear in mind that around 40 millennia passed between the time of the first known tally sticks and knotted cords and the first archeological records of mathematics as a craft. A craft that takes millennia to develop can hardly be demeaned as "trivial" or "obvious." Indeed, "arithmetic" was regarded in colonial America as an area of difficult scholarship. Cubberley tells us,

> A textbook was seldom used in teaching arithmetic by the colonial schoolmasters. The study itself was common but not universal. It was not until the beginning of our national period that arithmetic was anywhere made a required subject of instruction. The subject was regarded as one of much difficulty, and one in which few teachers were competent to give instruction, or few pupils competent to understand. To possess a reputation as an "arithmeticker" was an important recommendation for a teacher, while for a pupil to be able to do sums in arithmetic was a matter of much pride to parents. . . . It was not until the middle of the eighteenth century that printed arithmetics [arithmetic textbooks] came into use, and then only in the larger towns. [Cubberley (1919), pp. 32-33]

Well before Euclid's time and all the way down to today, the fact that measurements are implicit in mathematical statements was taken for granted and passed over with hardly a second (or even a first) thought. The hieroglyphic-like symbolic notation most people associate with mathematics today only served to reinforce neglect of the fundamental role measurements have in using mathematics to describe nature. This symbolism was much later in arriving on the scene, and its arrival was largely due to Descartes and his invention of "analytic geometry" around 1629-1633 [Smith (1923), pp. 374-375]. Until then, mathematical treatises consisted mainly of lengthy (and often not very clear) verbal statements sprinkled with some abstract symbols. One sees this in Euclid's *Elements*, Aristotle's *Prior Analytics* and *Posterior Analytics*, and the works of Kepler, Galileo, and Newton. That a present day introductory physics textbook tends to be arguably less verbose than science treatises of the 18th century and earlier is due mainly to its liberal use of mathematical equations that replace much of the earlier lengthy verbal descriptions. Yet the fundamental role of measurements in natural science remains.

To gain some better insight into this role and its epistemological consequences, try the following simple experiment. Using a straightedge, carefully draw a right triangle having a base 2 inches long and a height of 3 inches. Next take a ruler and very carefully measure the length of its hypotenuse. Depending on the ruler's scale marks (32nds of an inch, 64ths of an inch, 10ths of an inch, etc.), you will find that you get a result similar to

$3 + 38/64$ inches $<$ length of hypotenuse $< 3 + 39/64$ inches, or
3.60 inches $<$ length of hypotenuse < 3.61 inches

or some other similar result depending on the gradations etched into your ruler. The actual length of the hypotenuse falls *between* two scale markings. No ruler exists anywhere on earth that you can use to exactly measure the length and get the result that Pythagoras' Theorem tells us, namely $\sqrt{13}$ = 3.6055512… (where the "…" denotes a never ending and never repeating sequence of additional digits). In fact, $\sqrt{13}$ is an irrational number and cannot be written out exactly with a finite number of digits.

What you *actually know from experience* is only "3 + 38/64 inches < length of hypotenuse < 3 + 39/64 inches" (or whatever intervals your own ruler marks out). Granted the interval of uncertainty, 3 + 38/64 = 3.59375 < 3.6055512… < 3 + 39/64 = 3.609375 isn't very much and most of the time makes no *practical* difference in the real world of human activity. If a cellphone tower is 200 ft. plus 0.0625 inches tall instead of exactly 200 ft. tall, no one in its service area will notice any difference in service as a result.

But sometimes measurement uncertainties *do* make a difference and sooner or later cause problems for theoretical science. Recall, for instance, the Feynman quote from page 3. The fact that mathematical entities such as $\sqrt{13}$ and possible measurements in the real world are "not the same things" sometimes gives rise to baffling disagreements between theory and actual experience that are called ***paradoxes***. In chapter 2 we will look at an important epistemological principle called "Slepian's Principle," proposed in 1974 by a renowned mathematician named David Slepian, that will turn out to be very important for understanding how to properly use mathematics to describe empirical nature.

Feynman told his students,

> Although it is interesting and worth while to study the physical laws simply because they help us to understand and use nature, one ought to stop every once in a while and think, "What do they really mean?" The meaning of any statement is a subject that has interested and troubled philosophers from time immemorial, and the meaning of physical laws is even more interesting because it is generally believed that these laws represent some kind of knowledge. . . .
>
> Let us ask, "What is the meaning of the physical laws of Newton, which we write as $F = ma$? What is the meaning of force, mass, and acceleration?" Well, we can intuitively sense the meaning of mass, and we can *define* acceleration if we know the meaning of position and time. We shall not discuss those meanings, but shall concentrate on the new concept of *force*. The answer is equally simple: "If a body is accelerating, then there is a force on it." That is what Newton's laws say . . . [This] Newtonian statement . . . seems to be a most precise definition of force, and one that appeals to mathematicians; nevertheless, it is completely useless because no prediction whatsoever can be made from a definition. One might sit in an armchair all day long and define words at will, but to find out what happens when two balls push against each other, or when a weight is hung on a spring, is another matter altogether, because the way the bodies *behave* is something completely outside any choice of definitions. . . . If you insist on a precise definition of force, you will never get it! First, because Newton's Second Law is not exact, and second, because in order to understand physical laws you must understand that they are all some kind of approximation. . . . The trick is in the idealizations. [Feynman, *et al*. (1964), vol. I, chap. 12, pp. 1-2]

He went on to say,

> To begin with a particular force, let us consider the drag on an airplane flying through the air. What is the law for that force? . . . [It] is a remarkable fact that the drag force on an airplane is approximately a constant times the square of the velocity, or $F \sim cv^2$.
>
> Now what is the status of such a law, is it analogous to $F = ma$? Not at all, because in the first place this law is an empirical thing that is obtained roughly by tests in a wind tunnel. You say, "Well $F = ma$ might be empirical too." That is not the reason that there is a difference. The difference is not that it is empirical, but that, as we understand nature, this law is the result of an enormous complexity of events and is not, fundamentally, a simple thing. If we continue to study it more and more, measuring more and more accurately, the law will continue to become *more* complicated, not *less*. In other words, as we study this law of the drag on an airplane more and more closely, we find out that it is "falser" and "falser," and the more accurately we measure, the

> more complicated the truth becomes [*ibid*., pg. 3]

Whenever we speak of our knowledge of something being "approximate" what we are saying is that there is still uncertainty in the degree of truth and the degree of falsity our statements of this knowledge contain. Kant wrote,

> The opposite of truth is *falsehood*, which, insofar as it is taken for truth, is called *error*. An erroneous judgment – for there is error as well as truth only in judgment – is thus one that confuses the illusion of truth with truth itself. . . . Every error into which human understanding can fall is, however, only *partial*, and in every erroneous judgment there must always lie something true. For a *total* error would be a complete *opposition* to the laws of understanding and reason. . . . In respect to the true and the erroneous in our knowledge, we distinguish an *exact* cognition from a *rough* one. Cognition is *exact* when it is adequate to its Object, or when there is not the slightest error in regard to its Object, and it is *rough* when there can be errors in it yet without being a hindrance to its aim. [Kant (1800) 9: 53-54]

One might even say science is a quest to smooth out rough knowledge and make its cognitions of Objects exact enough for human purposes and aims. In his own way, this also is what Feynman was saying.

Some people find the idea that "numbers" are outcomes of measurements – and, therefore, that there is some amount of uncertainty involved even in them – profoundly disturbing. After all, every application of science and engineering eventually comes down to a verdict delivered by means of numbers. Some kinds of numbers – specifically the finite integers – seem to be inherently more exact than others – e.g., those we call the "real numbers" – if for no other reason than that, in principle, we can count them using our fingers. Counting is one kind of measurement and it's hard to see "what could go wrong" with counting.[6]

The situation becomes a little more uncertain when we get our numbers by means of other measurement methods such as counting the number of rods of length and width in an acre[7]. This is because the quantity we are interested in – the size of an acre – is determined "indirectly": we lay down a measuring rod over and over and count (e.g. with our fingers) how many times we laid it down. However, the measuring rod is itself a physical object and how are you to know the measuring rod *I* use is "the same length" as the measuring rod *you* use? That uncertainty is why, for instance, the ancient Phoenicians employed the idea of "standard weights and measures." It is also why governments establish agencies such as the National Institute of Standards and Technology (formerly known as the National Bureau of Standards, a part of the U.S. Department of Commerce).

Ignoring or neglecting the fact that such a basic idea as "numbers" is *practically* defined by means of measurement can and occasionally does lead to serious problems in science. Science historian Thomas Kuhn wrote,

> Normal science, the activity in which most scientists inevitably spend almost all their time, is predicated on the assumption that the scientific community knows what the world is like. Much of the success of the enterprise derives from the community's willingness to defend that assumption, if necessary at considerable cost. Normal science, for example, often suppresses fundamental novelties because they are necessarily subversive of its basic commitments. Nevertheless, so long as those commitments retain an element of the arbitrary, the very nature of normal research ensures that novelty shall not be suppressed for very long. Sometimes a normal problem, one that ought to be solvable by known rules and procedures, resists the reiterated onslaught of the ablest members of the group within whose competence it falls. On other occasions a piece of equipment designed and constructed for the purpose of normal research fails to perform in the anticipated manner, revealing an anomaly that cannot, despite repeated effort, be aligned with professional

[6] If you ask the U.S. Census Bureau "what could go wrong with counting?" you'd get a very lengthy answer.
[7] 1 rod = 16.5 ft. 1 acre = 43,560 square feet = 160 square rods.

> expectation. In these and other ways besides, normal science repeatedly goes astray. And when it does – when, that is, the profession can no longer evade anomalies that subvert the existing traditions of scientific practice – then begin the extraordinary investigations that lead the profession at last to a new set of commitments, a new basis for the practice of science. The extra-ordinary episodes in which that shift of professional commitments occurs are the ones known in this essay as scientific revolutions. They are the tradition-shattering complements to the tradition-bound activity of normal science. [Kuhn (1970), pp. 5-6]

One of the woefully unappreciated sources of, as Kuhn put it, science repeatedly going astray are the unexamined and ignored metaphysics of its practitioners – simply the unexamined "way in which one looks at the world." In the long history of science this has, over and over, meant ontology-centered meta-physics. Not to put too fine a point on it, but "the world looks very different" when it is viewed through the lens of an epistemology-centered metaphysic. In the chapters that follow, we will explore this very thing with particular attention to the use of mathematics in our descriptions of empirical knowledge.

References

Author's website for citations: http://www.mrc.uidaho.edu/~rwells/techdocs/

Bail, Ann (2003), *Encyclopedia of Catholic Devotions and Practices*, Huntington, IN: Our Sunday Visitor, pp. 485-487, ISBN 0-87973-910-X

Clapham, Christopher (1996), *Oxford Concise Dictionary of Mathematics*, 2nd ed., Oxford, UK: The Oxford University Press.

Cubberley, Ellwood Patterson (1919), *Public Education in the United States: A Study and Interpretation of American Educational History*, Boston, MA: Houghton Mifflin Co.

Davis, Philip J. & Reuben Hersch (1981), *The Mathematical Experience*, Boston, MA: Houghton Mifflin Co.

d'Errico, Francesco *et al*. (2012), "Early evidence of San material culture represented by organic artifacts from Border Cave, South Africa," *Proc. National Academy of Science* 109 (33) 13214-13219.

Feynman, Richard P. (1965), *The Character of Physical Law*, Cambridge, MA: The MIT Press, 21st printing, 1994.

Feynman, Richard P., Robert B. Leighton and Matthew Sands (1964), *The Feynman Lectures on Physics*, in three volumes, Reading, MA: Addison-Wesley Publishing Co.

Gödel, Kurt (1931), *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, NY: Dover Publications, 1992.

Haxton, Brooks (2001), *Heraclitus Fragments*, London: Penguin Classics.

Kant, Immanuel (early 1770s), *Logik Blomberg*, in *Kant's gesammelte Schriften*, *Band XXIV*, pp. 7-301, Berlin: Walter de Gruyter & Co., 1966.

Kant, Immanuel (1776-95), *Reflexionen zur Metaphysik*, 2nd part, in *Kant's gesammelte Schriften*, *Band XVIII*, pp. 3-725, Berlin: Walter de Gruyter & Co., 1928.

Kant, Immanuel (1783), *Prolegomena zu einer jeden künftigen Metaphysik, die als Wissenschaft wird auftreten können*, in *Kant's gesammelte Schriften, Band IV*, pp. 253-383, Berlin: Druck und Verlag von Georg Reimer, 1911.

Kant, Immanuel (1786), *Metaphysische Anfangsgründe der Naturwissenschaft*, in *Kant's gesammelte Schriften, Band IV*, pp. 465-565, Berlin: Druck und Verlag von Georg Reimer, 1911.

Kant, Immanuel (1787), *Kritik der reinen Vernunft*, 2nd ed., in *Kant's gesammelte Schriften*, *Band III*, Berlin: Druck und Verlag von Georg Reimer, 1911.

Kant, Immanuel (*c*. 1790-91), *Metaphysik* L_2, in *Kant's gesammelte Schriften*, *Band XXVIII*, pp. 531-610, Berlin: Walter de Gruyter & Co., 1970.

Kant, Immanuel (1800), *Logik*, in *Kant's gesammelte Schriften*, *Band IX*, pp. 1-150, Berlin: Walter de Gruyter & Co., 1923.

Kuhn, Thomas S. (1970), *The Structure of Scientific Revolutions*, 2nd edition (enlarged), Chicago, IL: The University of Chicago Press.

Leonard, William F. and Thomas L. Martin, Jr. (1980), *Electronic Structure and Transport Properties of Crystals*, Huntington, NY: Robert E. Krieger Publishing Co.

Nelson, David (2003), *The Penguin Dictionary of Mathematics*, 3rd ed., London: Penguin Books.

Newton, Isaac (1687), *Mathematical Principles of Natural Philosophy*, Seattle, WA: printed by CreateSpace, a division of Amazon (2010).

Newton, Isaac (1704), *Opticks*, Fourth London edition (1730), NY: Dover Publications, 1952.

Nicomachus (*c*. 100 AD), *Introduction to Arithmetic*, in *Great Books of the Western World*, vol. 11, pp. 805-848, Chicago, IL: Encyclopaedia Britannica, Inc., 1952.

Piaget, Jean (1929), *The Child's Conception of the World*, Savage, MD: Littlefield Adams, 1951.

Piaget, Jean (1941), *The Child's Conception of Number*, London: Routledge & Kegan Paul, 1952.

Piaget, Jean (1970), *Genetic Epistemology*, NY: W.W. Norton & Co., 1971.

Piaget, Jean and Rolando Garcia (1987), *Toward A Logic of Meanings*, Hillsdale, NJ: Lawrence Erlbaum Associates, 1991.

Plato (date uncertain), *Republic*, *c*. 4th century BC. Numerous editions available.

Russell, Bertrand (1919), *Introduction to Mathematical Philosophy*, London: Routledge, 1998.

Smith, David E. (1923), *History of Mathematics*, vol. I, NY: Dover Publications, 1958.

Suppes, Patrick (1972), *Axiomatic Set Theory*, NY: Dover Publications.

Thomas, Ivor (1939), *Greek Mathematical Works*, in two volumes, Cambridge, MA: Harvard University Press, 1991.

Wells, Richard B. (2006), *The Critical Philosophy and the Phenomenon of Mind*, available free of charge from the author's web site.

Wells, Richard B. (2009), *The Principles of Mental Physics*, available free of charge from the author's web site.

Wells, Richard B. (2012), *The Idea of the Social Contract*, 2nd ed., available free of charge on the author's web site.