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Chapter 5 Mathematics and Aesthetical Judgment

§ 1. Mathematics and Aesthetic

At the close of chapter 4, I proposed for your consideration the idea that definition and construction of mathematical structures aims to find ways to take already-familiar ideas and models and adapt them to the study of new things or the study of old things in new ways. That human beings are capable of doing this is undisputed. This is obvious when one considers that *invention* – whether in engineering, science, craftsmanship, poetry, imaginative literature, music, or visual arts – is a manifestation of doing precisely this. The word "invent" derives from the Latin word *invenire*, "to come upon, to find (something lost, hidden, etc.)." Childhood is the time of life when people are at their most inventive, although adults often belittle childish inventing by calling it "romancing" and fail to appreciate its developmental importance.

To give one charming example of childish invention, let us take a brief look at Piaget's observation of the behavior of a 3-year-old child upon her early encounters with marbles. After first describing his observations in detail, Piaget commented,

> The child is undoubtedly trying first and foremost to understand the nature of marbles and to adapt its motor schemes to this novel reality. This is why it tries one experiment after another: throwing them, heaping them into pyramids or nests, letting them drop, making them bounce, etc. But once it has got over the first moments of astonishment, the game still remains incoherent, or rather still subject to the whim of the moment. On days when the child plays at cooking dinner, the marbles serve as food to be stewed in a pot. On days when it is interested in classifying and arranging, the marbles are put in heaps in the holes of armchairs, and so on. . . . [It] is important to note the symbolism that immediately becomes grafted upon the child's motor schemes. These symbols are undoubtedly enacted in play rather than thought out, but they imply a certain amount of imagination: the marbles are food to be cooked, eggs in a nest, etc. [Piaget (1932), pp. 30-31]

Note how the child takes things she is already familiar with (motor schemes, watching her mother cook, the sight of bird nests) and makes her new playthings (the marbles) symbolic of these other things – an innovation for which *the possibility* is not immediately obvious from her "experiments" with throwing, dropping, and bouncing the marbles. More immediately germane to our present discussion, though, is this: from the moment she begins using the marbles as *symbols* for something else, in her imagination she is passing from the "concrete world" of Facet A into a "romantic world" of Facet B and is engaging in what is, for all practical purposes, a very crude but foundational form of mathematics. Remember that "mathematics" is a word derived from the Greek μαθημα (*mathema*) – "that which is learned; lesson." It does not matter that her learning proceeds through "manipulative" or "mechanical" actions; when *you* do calculations with an abacus or an electronic calculator, you are doing mathematics through manipulation too. Only your *symbolism* and your *purpose* differ from those of this little child. In such child's play we can see buds of mathematical ability manifested. Whether they blossom later is another matter.

The presence of a capacity for mathematical thinking *in every human being* is manifested in such humble ways; but what is the basis for it in the human phenomenon of mind? Why is math the way it is? These are the questions explored in this chapter.

Kant wrote that mathematics is knowledge through the construction of concepts. Epistemologically, concepts are rules (for the reproduction of intuitions), and so the possibility of mathematics is bound up inextricably with the capacities for cognition and construction of rules. This capacity in general is called *judgment*. He wrote,

> If understanding in general is explained as the faculty of rules, then the power of judgment is the capacity to subsume under rules, i.e., to distinguish whether something stands under a given rule or not. [Kant (1787) B:171]

A rule is an assertion made under a general condition. In the phenomenon of mind a Critical distinction is drawn between three kinds of general conditions for the rule making: practical; subjective; and objective. In his logic courses, Kant taught,

> The power of judgment is of two kinds: the *determining* or the *reflective* power of judgment. The first goes *from the general to the particular*, the second *from the particular to the general*. The latter has only *subjective* validity, for the general to which it proceeds from the particular is only *empirical* generality [Kant (1800) 9: 131-132].

It would be natural to presume that mathematical judgments would fall under the objective condition, especially since the constructing and structuring of mathematics involves the invention of mathematical objects. However, to so presume is to ignore the fact that the important Facet B objects – indeed, all mathematical objects – are generalizations of special cases. Generalizing judgments have only subjective validity. Objects of mathematics are invented and, in all cases, are supersensible objects.

In regard to the making of concepts, the two kinds of powers of judgment are logically divided into a capacity for *determining judgment* and a capacity for *reflective judgment* [Wells (2009)]. Both of these capacities deal with *perception*, i.e., representation with consciousness[1].

A general (that is, higher) concept is made by abstracting from lower concepts that it coordinates. Its making is adjudicated by the process of reflective judgment. However, reflective judgment is a *subjective* capacity of mind, i.e., a judgment of *how perception affects the state of mind* of the person making the judgment. This is something different from, and necessarily antecedent to, *understanding* perceptions objectively (i.e., re-cognizing them as cognitions of objects). The way Kant put it was

> If, however, I investigate more closely the interrelation of given cognitions in every judgment, and distinguish it as something belonging to understanding, from relationship in accordance with the laws of reproductive imagination (which has only subjective validity), then I find a judgment is nothing other than the way to bring given cognitions to the *objective* unity of apperception. [Kant (1787) B: 141]

Every concept begins as an intuition, and every intuition is synthesized in imagination before it is *re*-cognized as a representation of a concept [Wells (2009), chap. 3]. Concepts are rules for the *reproduction* of intuitions. Cognitions (objective perceptions with consciousness) are the representations by which human beings understand objects, but they *originate* from aesthetical representations of perceptions (affective perceptions) affecting the person's state of mind. The process of reflective judgment is the adjudicating process for the latter. The Critical science of the laws of sensibility is called Aesthetic and aesthetical representations of sensibility are adjudicated by the process of reflective judgment [Wells (2006), chap. 14].

The functions governing the process of determining judgment[2] are, so to speak, "local laws" because they are concerned only with cognitions of particular objects. But they leave undetermined the manner and way in which we understand objects in nature *as a system*. This is what Kant means by an "objective *unity* of apperception." That unification is, among other things, what the process of reflective judgment provides. Kant tells us

> Thus we must think of there being in nature, with regard to its merely empirical laws, a possibility of infinitely manifold empirical laws, which so far as our insight goes are nevertheless contingent

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[1] Motor rules (that is, sensorimotor schemes) are unconscious representations because, as modern psychology confirms, we are not conscious of the rules for *how* we perform any movement of locomotion. Kant's twofold logical division applies to sensorimotor schemes, too, but places the capacities for making them in two other *practical* capacities of mind. These are called, respectively, "practical judgment" and "appetitive power" [Wells (2009)].

[2] These functions are named "the categories of understanding."

> (cannot be recognized *a priori*), and with regard to them we bejudge the overall unity of nature as contingent in accordance with empirical laws and the possibility of the unity of experience (as a system in accordance with empirical laws). But because such a unity must still necessarily be presupposed and assumed, for otherwise no thoroughgoing context of empirical knowledge into a whole of experience would take place, because the general natural laws yield such a context among things with respect to their genus, as things of nature in general, but not specifically as such and such particular beings in nature, the power of judgment must thus take it as an *a priori* principle for its own use that what is contingent for human insight in the particular (empirical) natural laws nevertheless contains a lawful unity . . . in the combination of its manifold into one possible self-contained experience. Consequently, because the lawful unity in a combination that we recognize as conformable with a necessary aim (a requirement) but yet at the same time as contingent in itself, is represented as an expedience of the Objects (in this case, of nature), thus the power of judgment, which, having regard to things under possible (still to be discovered) empirical laws is merely reflecting, must think of nature with regard to the latter according to a *principle of expedience* for our faculty of knowledge [Kant (1790) 5: 183-184].

Kant's exasperating run-on sentences make this quite a mouthful, but there are two key points in here. First, human *systematic* understanding of nature and experience is not possible *without* the process of reflective judgment. Second, reflective judgment is governed by a principle of *expedience*. Expedience[3] is a property of a representation regarded as possible only with respect to some purpose from the practical Standpoint. The expedience of something is the congruence of a thing with that property of things that is possible only in accordance with purposes. But purposes belong to the human being who is doing the judging – and therefore expedience is subjective in its practical aims.

It follows from all this that for an understanding of the fundamental nature of mathematical thinking we must – perhaps surprisingly – turn to what can be called a Critical metaphysic of psychology for its explanation and theory. Kant was able to bring Critical metaphysics right up to the edge of such a metaphysic but was unable to provide us with one because in Kant's day no *empirical science* of psychology existed. Without such a science he had few sources of particular concepts from which to generalize[4]. Philosophical knowledge is rational cognition from concepts [Kant (1787) B: 741] and so to make a scientific metaphysic for Aesthetic and reflective judgment one must have scientific concepts to work with. Psychology does not *provide* a metaphysic but its observations and experiments do provide raw material for it. What Piaget said of his "genetic epistemology" applies equally well to a Critical metaphysic of Aesthetic:

> What I have said so far may suggest that it can be helpful to make use of psychological data when we are considering the nature of knowledge. I should like now to say that it is more than helpful; it is indispensable. In fact, all epistemologists refer to psychological factors in their analyses, but for the most part their references to psychology are speculative and are not based on psychological research. I am convinced that all epistemology brings up factual problems as well as formal ones, and once factual problems are encountered, psychological findings become relevant and should be taken into account. [Piaget (1970), pp. 7-8]

In regard to Aesthetic, the psychological findings most pertinent are those dealing with such topics as "emotion" and, generally, subjectivity, affectivity and intuition.

§2. Intuition and Objectivity

Intuition, "emotion," and affectivity are topics that have a rather checkered history in psychology and,

[3] In German, *Zweckmäßigkeit*.
[4] While various philosophers opined about psychology as far back as the ancient Greeks, the beginning of psychology as an empirical science is generally credited to Wilhelm Wundt beginning in 1874 – seventy years after Kant died.

for the most part, have been approached through ontology-centered ways of looking at the world. So far as I know, a *systematic* Critical treatment of these topics first appeared only in 2006 [Wells (2006)]. Mathematics is knowledge through the construction of concepts; all concepts begin as intuitions; and the judgment of every intuition is adjudicated by the *subjective* process of reflective judgment. Accordingly, an understanding of the human capacity for mathematics must seek this understanding in Critical Aesthetic and the process of reflective judgment.

There are some people – including many mathematicians – who object to the idea that mathematical knowledge has its origins in intuition (much less, in affectivity). The Bourbaki, for example, were strenuous in their objections to the point where the entire Bourbaki movement in mathematics can be seen as a revolt against the "intuitionist" theme put forth by Poincaré [Poincaré (1914), pp. 46-63] that dominated education in their day. Their objection, however, is predicated on the unsatisfactory vagueness of the commonplace meanings given to the word "intuition." In Critical epistemology, the term "intuition" has a very specific meaning, and this meaning is not how the Bourbaki understood what "intuition" is.

The Bourbaki viewed mathematics as a rigorous discipline (in the connotation of "rigor" as "strict precision"), and saw rigor and intuition as incompatible opposites. But, as I shall explain in this chapter, rigor in representation does not stand in opposition to Critical intuition and precision is inherent in reflective judgments that bring forth mathematical intuitions. Furthermore, even the Bourbaki did not entirely banish intuition from mathematics provided that it was "tamed" by analytic rigor and thereby taken out of what they saw as its "chaotic state" [Pier (undated)].

In Critical epistemology, intuition is the immediate reference of the power of representation to an individual Object [Kant (1776-95) 18: 282]. An intuition is a direct, singular and sensible objective perception of an appearance and is presented in sensibility. However, no representation in sensibility is objective (refers to an object) until and unless that representation is *marked* as such by an act of reflective judgment [Wells (2009), chap. 7, pp. 248-254], [Wells (2006), chap. 18].

Perhaps this Critical finding – that no representation in sensibility is objective until and unless reflective judgment declares it to be so – strikes you as strange or even, perhaps, nonsensical. It seems to have struck Aristotle, the British empiricists (Locke and Berkeley), and the continental rationalists (Descartes and Leibniz) so. Aristotle, Locke, and Berkeley thought that the data of sensation were "stamped into" or "impressed upon" the mind by external objects. The rationalists thought human beings were born with "innate ideas" of preformed knowledge of objects. Both philosophies made what I have elsewhere called a "copy of reality" hypothesis to explain the phenomenon of human objective perception [Wells (2006), chap. 3, pp. 187-189]. Kant, on the other hand, found no basis in fact for any copy of reality hypothesis and in recent years both psychology and neuroscience came to this same conclusion. The eminent neuroscientist Walter Jackson Freeman III of the University of California Berkeley wrote,

> Our brains don't take in information from the environment and store it like a camera or tape recorder for later retrieval. What we remember is continually being changed by new learning, when the connections between nerve cells in the brain are modified.
>
> A stimulus excites the sensory receptors so that they send a message to the brain. That input triggers a reaction by which the brain constructs a pattern of neural activity. The sensory activity that triggered the construction is then washed away, leaving only the construct. That pattern does not "represent" the stimulus. It constitutes the meaning of the stimulus for the person receiving it.
>
> That meaning is different for every person because it depends on their past experience. Since the sensory activity is washed away and only the construct is saved, the only knowledge that each of us has is what we construct in our own brains. [Freeman (1998)]

Years earlier, Piaget's research led him to the finding that

> Observation and experimentation combined seem to show that object concept, far from being

> innate or given ready-made in experience, is constructed little by little. Six stages can be discerned, corresponding to those of intellectual development in general. [Piaget (1954), pg. 4]

Both these quotations speak directly to the individuality – by another word, subjectivity – of human objective perceptions. Objective judgments of cognitions are adjudicated by the process of determining judgment, but this process must be *given* the cognitions upon which it operates. It cannot supply its own; if it could that would be a "copy of reality" mechanism and there is no such thing in human mentality. That leaves only the subjective process of reflective judgment as the adjudicating mental process for endowing a perception with the property of "objectivity."

And, indeed, the *synthesis of objectivity* is one of four major synthetic processes, participated in by the process of reflective judgment, at work in effecting a complete state of conscious representation [Wells (2009), chap. 7, pp. 238-241, 254-260]. Deduction and explanation of this aspect of the "mental physics" of the human phenomenon of mind is too lengthy to include here, but is provided in detail elsewhere [Wells (2006)], [Wells (2009)]. Here I wish to "zero in" on just one aspect of all this, *viz*., the aspect responsible for the possibility of mathematics and the mental nature of mathematical thinking.

As you might already anticipate, the process of reflective judgment must be a complex process because of the many important roles it has and exhibits in human psychology. Because of this complexity, Critical epistemology makes a logical division of reflective judgment into a process of *aesthetical* reflective judgment and a process of *teleological* reflective judgment. The latter serves as a metaphorical "bridge" between understanding and reasoning and also establishes the relationships between affective perceptions and motor actions a person physically expresses. Aesthetical reflective judgment, on the other hand, establishes the relationships between sense and feelings, on the one hand, and the subjective state of a person on the other. Figure 1 illustrates the logical organization of aesthetical reflective judgment and names the twelve fundamental functions contained in its process [Wells (2009), chap. 8]. As you can see, the four "headings" in this figure pertain to satisfactions, the making of transcendental judgments of affirmation and denial, values and interests, and functions of expedience for happiness. Overall, the process of aesthetical reflective judgment is held to be the process for the synthesis of desire.

Of the twelve elementary functions depicted in figure 1, one of them is central to the topic of the human nature of mathematics. This is the function called *the feeling of beauty*. If you are one of the great many people who have been taught to have an aversion to mathematics because of the way it is taught in schools [Wells (2014), chap. 14], it may seem very strange to associate "beauty" with mathematics. Most mathematicians and many scientists, on the other hand, testify to having this feeling about mathematics.

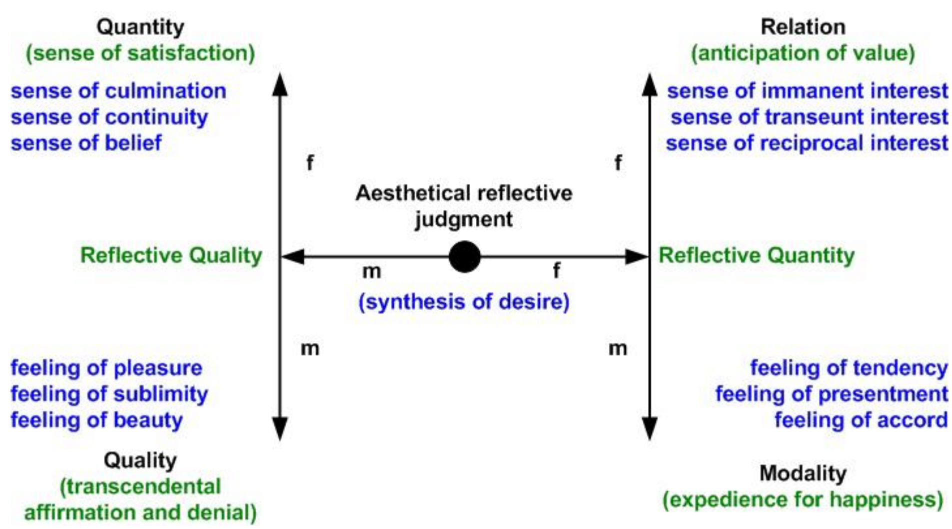

Figure 1: 2nd level analytic representation of the process of aesthetical reflective judgment. f = form; m = matter.

Poincaré, for example, spoke eloquently about this role in creative mathematical thinking. He wrote,

> More commonly the privileged unconscious phenomena, those that are capable of becoming conscious, are those which, directly or indirectly, most deeply affect our sensibility.
>
> It may appear surprising that sensibility should be introduced in connection with mathematical demonstrations, which, it would seem, can only interest the intellect. But not if we bear in mind the feeling of mathematical beauty, of the harmony of numbers and forms and of geometric elegance. It is a real aesthetic feeling that all true mathematicians recognize, and this is truly sensibility.
>
> Now, what are the mathematical entities to which we attribute this character of beauty and elegance, which are capable of developing in us a kind of aesthetic emotion? Those whose elements are harmoniously arranged so that the mind can, without effort, take in the whole without neglecting the details. This harmony is at once a satisfaction to our aesthetic requirements, and an assistance to the mind which it supports and guides. At the same time, by setting before our eyes a well-ordered whole, it gives us a presentment of a mathematical law. Now, as I have said above, the only mathematical facts worthy of retaining our attention and capable of being useful are those which can make us acquainted with a mathematical law. Accordingly, we arrive at the following conclusion. The useful combinations are precisely the most beautiful, I mean those that can most charm that special sensibility that all mathematicians know, but of which laymen are so ignorant that they are often tempted to smile at it. [Poincaré (1914), pp. 58-59]

When an aesthetic judgment of the feeling of beauty is rendered, Critical epistemology teaches that three other functions (one from each of the three remaining headings) are co-involved in the complete synthesis of a judgment of desire. This means there are $3^3 = 27$ *species* of aesthetical judgments that contain a judgment of the feeling of beauty. Collectively, these are called **judgments of taste**.

But, because aesthetical reflective judgment knows no Objects *as* objects, what is it that is being judged as a judgment of taste? What is its fundamental condition in the phenomenon of mind? Kant provided the answer to this question. Rendering of a judgment of taste denotes a subjective state of *harmonization in the free play of imagination and understanding* [Wells (2009), chap. 7, pg. 263], [Kant (*c*. 1773-79) 15: 424]. In Critical terminology, harmonization is *making diverse representations compatible and homogeneous with each other such that they can be combined in composition*. Does this not seem to be precisely the sort of thing Poincaré was describing above? Kant tells us,

> The judgment of taste differs from the logical [judgment] in that the latter subsumes a representation under concepts of the Object, but the former does not subsume under a concept at all, because otherwise the necessary universal approval could be compelled by proofs. All the same, however, it is similar to the latter in that it professes a universality and necessity, though not according to concepts of the Object and hence a merely subjective one. Now since the concepts in a judgment constitute its content (that which pertains to the cognition of the Object), but the judgment of taste is not determinable by means of concepts, it is grounded only on the subjective formal condition of a judgment in general . . . This, employed with respect to a representation through which an object is given, requires the harmonization of two powers of representation: namely of imagination (for the intuition and composition of its manifold), and of understanding (for the concept as representation of the unity of this composition). Now since no concept of the Object is here the ground of the judgment, it can subsist only in the subsumption of imagination itself (in the case of a representation through which an object is given) under the condition that understanding in general advance from intuitions to concepts. I.e., because the freedom of imagination subsists precisely in the fact that it schematizes without a concept, the judgment of taste must rest on a mere sensation of the reciprocally animating power of imagination in its *freedom* and understanding with its *conformity to law*, thus on a feeling that allows the object to be judged according to the expedience of the representation (through which an object is given) for the promotion of the faculty of knowledge in its free play; and taste, as a subjective power of judgment, contains a principle of subsumption, not of intuitions under *concepts* but rather of the

> power of imagination or presentations (i.e. of imagination) under the capacity for concepts (i.e. understanding) so far as the former *in its freedom* is in harmony with the latter *in its conformity to law*. [Kant (1790) 5: 286-287]

To know all this is already to know a great deal about the human nature of cognition. In Critical epistemology, imagination, in the narrow sense, is the ability to present an object in intuition through mental spontaneity. Understanding is the capacity for making a cognizant structure of rules by means of representations. Even so, this is still not quite enough to satisfy our purposes here. Crudely put, we need to know something about "how aesthetical taste works." What "goes into" the making of such a judgment and to what, concretely, does this sort of judgment speak? At first encounter, these are daunting questions. Fortunately, Kant also provided their answers. We find them in what he called "the approvals of taste."

§3. The Approvals of Taste

Out of countless possible representations in sensibility, a judgment of taste makes selections of those that are generally engaging according to laws of sensibility. Kant tells us,

> In everything that is to be approved in accordance with taste there must be something that facilitates the differentiation of the manifold (patterning); something that promotes intelligibility (relationships, proportions); something that makes the pulling of it together possible (unity); and finally, something that promotes its distinction from all other possibilities (precisioning). [Kant (*c*. 1773-79) 15: 271]

These four "somethings" are the synthetic functions of the approvals of taste. They are called *patterning*, *conceptualizing*, *coalescing*, and *precisioning*, respectively. In the terminology of Critical epistemology, these functions fall under the general headings of Quantity, Relation, Quality, and Modality, respectively [Wells (2006), chap. 3].

In the Critical doctrine, the term "faculty" means the form of an ability insofar as that ability is represented in an idea of organization. Mathematical ability is therefore properly thought of as a faculty of knowledge that is exhibited by judgments of taste. Viewed as such, mathematics is most immediately connected with the approvals of taste [Wells (2014), chap. 14]. To understand these four synthetic functions is therefore to understand the human nature of mathematical invention.

3.1 Patterning: ***Patterning*** is *the act of representing a pattern*. A ***pattern*** is *an arrangement of form as a grouping or distribution of elements*. In terms of judgments of taste, mathematics is the aesthetical determination of patterns found in the data of the senses. This gives us one property of mathematics in general: ***mathematics is all about patterns***. Now, the general idea of a "pattern" is very broad. In formal theoretical math the constructs called ***theorems*** are, at their judicial roots, ***nothing more and nothing less than identified patterns***. As an example, let us consider one of the simplest patterns in mathematics: the addition table (A plus B) for the decimal digits zero through 9 (figure 2). If you look along the top ($A = 0$) row of figure 2, you will see that this row is merely the regular counting sequence from 0 to 9. Now do the same for the first column ($B = 0$). You see the same sequence. This *pattern* in the table is the conceptual *root* of the idea of "the additive identity element," zero, in mathematics.

Now look at the $B = 1$ column. What you see is the counting sequence beginning with B. Look at the $B = 2$ column. You see the counting sequence *beginning with* B *once again*. This *pattern* is repeated in every column in the table. But there is another as well. When the count reaches "10" the non-carry digit "wraps around" to remain within the 0 to 9 range with the "wrap around" row going up by one box each time you move one column to the right. That's another pattern. This one provides the root idea for what mathematicians call an "equivalence class" and computer logic designers call a "rotate operation."

Some people say of the field of mathematics that it is "clear and inspiring." Certainly there are people who disagree, but I suggest that this disagreement stems from a failure of mathematics instruction. It isn't

A \ B	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	$^{1}0$
2	2	3	4	5	6	7	8	9	$^{1}0$	$^{1}1$
3	3	4	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$
4	4	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$
5	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$
6	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$
7	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$
8	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$	$^{1}7$
9	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$	$^{1}7$	$^{1}8$

^{1} denotes a carry

Figure 2: Base 10 addition table with carries indicated.

the axiom structure of mathematics nor its "logical purity" that makes mathematics "clear and inspiring." What makes math "clear and inspiring" are its patterns. Failure to see "clarity" in math, or for its teaching to be non-inspirational, comes from focusing on object cognition as the important thing in learning mathematics and not recognizing that the real importance lies in its patterns and the aesthetical function of patterning in an approval of taste.

There are still more patterns hiding in the addition table. Look down the $B = 9$ column. From $A = 1$ through the rest of the column, the non-carry digit is "A minus 1." Look down the $B = 8$ column; the non-carry digit is the digit in the $B = 9$ column minus 1. Look at the $B = 7$ column; you get the digit in the $B = 8$ column minus 1. The same "minus 1" idea holds as you go leftward in the table. *This is the pattern of a pattern.* If you know about the first pattern and *its* pattern, you can learn to do addition in your head and there is no need to try to memorize the entire table.

If you make a game of looking for them, you might be surprised by how many patterns you can discover in the dull old addition table. For example, look at the right to down-and-left minor diagonals entries. Mathematicians call this "symmetry" and it underlies, among other things, the commutative property of addition. You can also come to the idea of the commutative property by observing that the same "game" I described above for the columns of the table also works for the rows. This is a pattern to the pattern of the patterns. There is a lot of mathematics, going well beyond basic arithmetic, latent in the addition table. For example, start anywhere in the top row and go down the diagonal left-to-right (wrapping about to the left side of the table after you reach the right-hand edge). You'll see that this is "counting by twos." All of mathematics consists of patterns and patterns of patterns. Patterns please aesthetically because, once recognized, they can be *anticipated* and so are *aesthetically expedient*.

The ideas of patterns and patterning speak to something very fundamental in human psychology and the development of human intelligence. Almost from the very start of his decades of research, Piaget noted and observed *behavior patterns* in infants and young children, and he was able to build upon what he observed in these behavior patterns to erect what is arguably the most wide-ranging and self-consistent psychological theory of the development of human intelligence that has ever been put forth. Central to this theory, and coming directly out of behavior patterns, is the idea Piaget called a *scheme*:

> Whatever is repeatable and generalizable in an action is what I have called a scheme, and I maintain there is a logic of schemes. Any given scheme itself does not have a logical component, but schemes can be coordinated with one another, thus implying the general coordination of actions. These coordinations form a logic of actions that are the point of departure for the logical mathematical structures. [Piaget (1970), pg. 42]

I said earlier in this treatise that all meanings are *at root* practical. This is not only a requirement of Critical epistemology but, additionally, is a finding of fact coming out of the theory of schemes [Piaget & Garcia (1987)]. The first two years of human life are described by Piaget as "the stage of sensorimotor intelligence" [Piaget (1952)] and are characterized by the child's development of basic sensorimotor schemes of action and a gradually growing cognizance of these action schemes that give birth to *mental* schemes (in Piaget's words, "logico-mathematical schemes") of object recognition and problem solving. Indeed, the research of Piaget and his associates found that objective cognizance follows a path by which a child's awareness of his practical actions are "interiorized" to form understandings of mental structures [Piaget (1974)]. This is as much as to say determinant judgments of understanding are *products* of a synthesis (harmonization) of practical (non-conscious) judgments of actions and reflective judgments adjudicated as judgments of taste.

To recap the main point I wish to convey in this section: Discovery and invention in mathematics is, in terms of forms of mathematical composition, recognition of patterns and patterns of patterns observed in the object that is to be described by mathematics. In empirical science, to search for scientific explanation in mathematical form is to search for patterns exhibited by phenomena.

3.2 Coalescing: *Coalescing* is *the aesthetic function of syncretism in judgmentation*. Syncretism and juxtaposition are well known phenomena that have been studied in children [Piaget (1928), pp. 221-232]. Psychology research has taken note of them going back at least as far as the days of William James:

> Where the parts of an object have already been discerned, and each made the object of a special discriminative act, we can with difficulty feel the object again in its pristine unity; and so prominent may our consciousness of its composition be that we can hardly believe that it ever could have appeared undivided. But this is an erroneous view, the undeniable fact being that *any number of impressions, from any number of sensory sources, falling simultaneously on a mind* WHICH HAS NOT YET EXPERIENCED THEM SEPARATELY, *will fuse into a single undivided object for that mind*. The law is that all things fuse that *can* fuse, and nothing separates except what must. [James (1890), vol. I, pp. 487-488]

The capacity for *objective* syncretism ('fusion') is built into the synthesis of representational matter in sensibility to compose of an intuition. The formation of an intuition, in terms of its matter, is a process that can be pictured as a sort of accretion process in which divers sensible representational matter coalesces ('fuses') into a singular representation. Kant called this process *Gestaltung* ("formation"). Sensibility does not judge and *Gestaltung* of an intuition is adjudicated by reflective judgment. Of particular interest for the present topic-at-hand is the "flavor" of this approval of taste in regard to the process of reflexion. Here, Kant tells us,

> Reflexion (*reflexio*) does not have to do with objects themselves, in order to acquire concepts directly from them, but is rather the state of mind in which we first prepare ourselves to find out the subjective conditions under which we can arrive at concepts. It is the consciousness of the relationship of given representations to our various sources of knowledge through which alone their relationships among themselves can be directly determined. [Kant (1787) B: 316]

The coalition that occurs during *Gestaltung* of an intuition is an objective coalition, but the coalescing function of taste by which representation is adjudicated is a subjective function.

Coalescing is an agreement function and so immediately pertains to mathematics. Instruction in regard to this Quality function does not so much pertain to stimulating coalescing because the learner's laws of sensibility and judgmentation will perform coalition as a necessary part of the way the phenomenon of mind works. The aim of instruction, rather, is to try to get the learner's coalition to coalesce around an *intended* object of intuition at which the instruction is aimed. This is to say that the instruction is aimed at setting up a state of mind in the learner such that reflexion gathers together that which pertains to the intended lesson. To put it another way, Quality in mathematics instruction has the task of *combating the phenomenon of juxtaposition* rather than stimulating the phenomenon of syncretism.

Here James' remark quoted above is especially pertinent. Bearing in mind that theoretical math achieves knowledge through generalizing constructions of concepts, Quality in math instruction must be concerned with gathering together previously learned mathematical concepts and seeking to integrate these in a concept structure of greater generality. This means *de facto* that divers objects are to be brought together and seen (by the learner) as parts of some greater mathematical object. There is a great difference between merely "getting an answer" and "getting a principle for getting answers." Too much of the traditional instruction in mathematics is aimed at "getting an answer" and too little at "getting a principle."

To illustrate what I mean, consider the progression of arithmetic → algebra → geometry → trigonometry → analytic geometry → calculus. Each prior step in this progression is necessary for each succeeding one. This is shown by nothing more complicated than just looking at what goes into the increasingly more complex operations involved in each of these topics. I do not mean by this that *everything* pertaining to algebra is propaedeutic to *everything* in geometry – that is obviously not so – but, rather, that *some* concepts of algebra are needed by geometry, etc. For example, the proof of proposition 6 in Book III of Euclid's *Elements*, "If two circles touch one another, they will not have the same center," calls upon concepts of what we today call algebra. This is, of course, the reason that these courses are presented in the order they are presented in junior high school and high school mathematics curricula.

But the very fact that the divers parts of mathematics *are* taught in a sequence, and that each has objects peculiar to it, means that at the next step the learner is confronted by objects he has already experienced in prior classes as well as new ones being introduced to him at that moment. He therefore is psychologically confronted by a juxtaposition of objects he must learn to 'fuse together' in a mathematical object of greater scope of generality. His challenge is not to "feel the object again in its pristine unity" (as James put it) but rather to come to know a new (and supersensible) object *for the first time*.

Juxtaposition is the cognitive tendency to regard object 1 as "separate from but going with" object 2. How can a person combat the phenomenon of juxtaposition? To answer this it is important to understand how reflexion is adjudicated by reflective judgment. The process of reflexion in the synthesis of sensibility is a species of comparison. But in this process representations are not compared *to each other* but, rather, *to the state of the thinking/reasoning Subject* (the person) in regard to the compatibility of the representation with expedience for pure Reason. There are three ways representation in sensibility can be expedient for a person [Wells (2009), chap. 3, § 4.2]. All three involve harmonization, i.e., the act of making diverse representations compatible and homogeneous with one another so that they may be combined in composition. The first is harmonization of the free play of imagination and understanding, and this is equilibrium in *thinking*. The second is harmonization in the interaction between sensibility and reflective judgment, and this is called aesthetic harmony. The third is equilibrium in the cycle of judgmentation, and this is called the harmonization of reasoning. From a judicial Standpoint, ***reflexion*** is *the act of coalition in sensibility that produces any of these three forms of harmony*. Mathematically, reflexion is a synthesis that produces what mathematicians call "congruence classes."

Representations in sensibility of juxtaposed objects that are able to be harmonized by reflexion can for this reason be brought into a unity of representation under a new concept that understands them all by means of polysyllogism structures. For example, you are familiar with the idea of 'distance' as 'how far apart two things are.' But 'how far apart' is a concept with a lot of different meanings *depending on context*. The one most people use most of the time is what mathematics calls "Euclidean distance." Its formula is that of the familiar Pythagorean theorem, "the square root of the sum of the squares of two sides of a right triangle equals the hypotenuse." But 'how far apart' might also mean "two days' walk from here." Or it might mean "how many letters are different in two words" (this is called a Hamming distance; the words 'cat' and 'cad' are a Hamming distance of 1 from each other because they differ in just one letter; they are "one letter apart in spelling"). Mathematics unifies *all* these different ideas of "distance" under one concept, which is called a "metric" and is used to define a "metric space."

Let us take a keen look at these examples to see what is going on and in what way mathematics can be said to "harmonize" these different versions of "distance." Each of these ideas of 'distance' has a specific *context* in which it "makes sense" and each does not "make sense" when we try to use it in one of the *other* contexts. It is nonsense to say 'cat' is "one days' walk" from 'cad' (unless 'cat' and 'cad' are the names of two towns), and most people think it is nonsense to try to work 'cat' or 'cad' into the notion of triangles (although one can). The general concept of a 'metric,' however, provides the idea of *a general meaning implication schema* under which the divers specifying concepts of these different "distances" *can all be understood under one set of conditions*. It provides, in other words, a general definition of the notion of "distance" under which divers *species* of distance are understood as special cases. This is the significance of *adjectives* such as "Euclidean" distance or "Hamming" distance or "walking" distance.

And here we can find a concrete practical explanation for what "harmonizing different representations" means. The divers concepts *with* their contexts (Euclidean, Hamming, and walking distance) are brought together and "made free of" the *specialization* given to each by their contexts so that what is left after "skinning" one of them of its context is in some way *the same as what is left after "skinning" the others*. The "way they are the same" might involve some non-trivial practical scheme. But once the scheme has been worked out, then all the various meaning implications of the originally juxtaposed concepts are "the same despite being different."

This is not really as obscure a thing as my examples here might make it seem. You do these kinds of comparisons of reflexion every day. Barbara Bush was a human being. So are you. So was Albert Einstein. Obviously being Barbara Bush is not the same as being Albert Einstein but they are both, nonetheless, known to be human beings. They are "the same only different." And *that* is what the coalescing function of taste does: it makes juxtaposed things "the same only different." Cultivating the Quality of judgments of taste basically means developing maxims of *looking for how things can be the same even though they are different*, and making this act of looking-for-sameness *habitual*.

And, by the way, this idea of "things being the same only different" is a brief way of stating the idea of "what a number is" that was presented back in chapter 1, pages 12-13. It is why, in some chemistry contexts, a "molecule" can be a "funny number."

3.3 Conceptualizing: *Conceptualizing is the comprehension functional of reasoning*. In terms of its effects in reflective judgment, it is the *a priori* aesthetic functional of taste that refers an intuition to teleological reflective judgment by *signifying* the intuition is practically expedient for a purpose of practical Reason and is motivationally expedient for the manifold of concepts.

Kant called conceptualizing "the intelligibility function of approval of taste," but what is meant by saying this? What is 'intelligibility' to be taken to mean? 'Intelligibility' is an English rendering of the word Kant actually used, which was *Begreiflichkeit*. The verb *begreifen* means "to comprehend" and it denotes the highest of the seven levels of Kant's hierarchy classification of *degrees of perfection* to which one's knowledge can be raised [Kant (1800) 9: 64-65]. Kant's seven levels are:

1. <u>to represent</u> something;
2. to represent something with consciousness, i.e., <u>to perceive</u>;
3. <u>to be acquainted</u> with something, i.e., to represent something in comparison with other things as to sameness and to difference;

4. to be acquainted with something with consciousness, i.e., to cognize it;

5. to understand something, i.e., to know through understanding by means of concepts;

6. to know something through reason, i.e., to "have insight" into it;

7. to comprehend something, i.e., to know through reason to a degree that is sufficient to satisfy one's aim (that is, to satisfy a condition for making actual one's intended purpose).

Kant remarked that comprehension is only relative (sufficient for a specific aim), and that we do not comprehend *anything* without qualification. The *conceptualizing function* pertains to *on-going perfecting of one's knowledge* by raising it up through Kant's knowledge hierarchy. Perfect*ing* is a process.

Poincaré once recounted an anecdote in which he described how he had come to write his first treatise on his discovery of what mathematicians call "Fuchsian functions" [Poincaré (1914), pp. 52-57]. This is an advanced topic in pure mathematics with applications to non-Euclidean geometry. Mathematicians regard Fuchsian functions as very important, or so I am told, but his idea of them is far too complicated to explain what they are in this treatise. The theory of them is one of the reasons Poincaré is a famous mathematician. The very difficulty of explaining his idea makes it suitable to serve as an example of comprehension at Kant's seventh level of knowledge perfection. In his account, Poincaré described, in nontechnical language, his step by step psychological experiences in developing his theory from 1880 to 1883[5]. In his lengthy anecdote, Poincaré made a comment that is pertinent to the discussion here:

> One is at once struck by these appearances of sudden illumination, obvious indications of a long course of previous unconscious work. The part played by this unconscious work in mathematical discovery seems to me indisputable, and we shall find traces of it in other cases where it is less evident. Often when a man is working at a difficult question, he accomplishes nothing the first time he sets to work. Then he takes more or less of a rest, and sits down again at his table. During the first half-hour he still finds nothing, and then all at once the decisive idea presents itself to his mind. We might say that the conscious work proved more fruitful because it was interrupted and the rest restored force and freshness to the mind. But it is more probable that the rest was occupied with unconscious work, and that the result of this work was soon afterwards revealed to the geometrician exactly as in the cases I have quoted, except that the revelation, instead of coming to light during a walk or a journey, came during a period of conscious work, but independently of that work, which at most only performs the unlocking process, as if it were the spur that excited into conscious form the results already acquired during the rest, which till then remained unconscious. [Poincaré (1914), pg. 55]

I call to your attention levels 1 through 4 in Kant's hierarchy above where he draws a distinction between representing or becoming acquainted with something vs. representing or becoming acquainted with something *with consciousness*. Perfecting one's knowledge is a *process* that is first carried on aesthetically – without consciousness of an object – before representation in sensibility is *marked* as an intuition (by reflective judgment) and re-cognized in concept form.

I can attest to having experienced for myself the sort of "illumination" process Poincaré describes, and I have heard testimonies by others of having had the same sort of experience. If a person is introspective to some conscious degree, how experiencing of some "flash of insight" comes about seems mysterious. But this is because "having insight" is already an advanced stage of knowledge perfection (level 6). A great deal of *aesthetic* formulation necessarily precedes cognizance of it. Santayana wrote,

> Intent is one of many evidences that the intellect's essence is practical. Intent is action in the sphere of thought; it corresponds to transition and derivation in the natural world. Analytic psychology is obliged to ignore intent, for it is obliged to regard it merely as a feeling; but while the feeling of intent is a fact like any other, intent itself is an aspiration, a passage, the recognition

[5] see https://histsci.fas.harvard.edu/files/hos/files/doran_poincares_path_to_uniformization_2018.pdf .

> of an object which not only is not a part of the feeling given but is often incapable of being a feeling or a fact at all. What happened to motion under the Eleatic analysis happens to intent under an anatomizing reflection. The parts do not contain the movement of transition which makes them a whole. Moral experience is not expressible in physical categories, because while you may give place and date for every feeling that something is important or is absurd, you cannot so express what these feelings have discovered and have wished to confide in you. Yet it is this pronouncement concerning what things are absurd or important that makes the intent of these judgments. . . .
>
> Feelings and ideas, when plucked and separately considered, do not retain the intent that made them cognitive or living; yet in their native medium they certainly lived and knew. If this ideality or transcendence seems a mystery, it is such only in the sense in which every initial or typical fact is mysterious. Every category would be unthinkable if it were not actually used. . . . The fact that intellect has intent, and does not constitute or contain what it envisages, is like the fact that time flows, that bodies gravitate, that experience is gathered, or that existence is suspended between being and not being. . . . Cognition, too, is an expedient for vanquishing instability. As reproduction circumvents mortality and preserves a semblance of permanence in the midst of change, so intent regards what is not yet, or not here, or what exists no longer. Thus the pulverization proper to existence is vanquished by thought, which in a moment announces or commemorates other moments, together with the manner of their approach or recession. . . . What renders the image cognitive is the intent that projects it and deputes it to be representative. It is cognitive only in its use, when it is the vehicle of an assurance which may be right or wrong because it takes something ulterior for its standard. [Santayana (1906), pp. 172-174]

Santayana describes intent by a circumambulation but this is because *affective* perceptions are themselves incommunicable through the *objective* vehicle of language. In Critical terminology, intent is the determination of an action expression (including acts of thinking) according to a rule or a maxim of practical Reason. The matter of intent is a feeling of subjective expedience, the form of intent is a determined practical appetite. An appetite is the representation of a practical purpose. The process of practical Reason knows no objects and feels no feelings but it does control the employment of understanding and does so according to a standard gauge of perfection [Wells (2009), chap. 12].

The conceptualizing functional of approval of taste *signifies* an intuition is expedient for a purpose and motivational for understanding, and thereby gives it its contexts. When understanding is achieved and presented in a resulting intuition, the making of that resulting intuition constitutes the "flash of insight" as Poincaré and others have described it. Expedience for a purpose, however, is not in and of itself the satisfaction of a purpose. Comprehension (level 7) occurs only when that purpose is sufficiently satisfied. Thus "insight" is followed up by additional acts of reasoning ("comprehending") that bring closure to the process overall. This is what is meant by saying the conceptualizing functional "promotes" intelligibility. It is called a "functional" because the conceptualizing approval of taste does not consist of just *one* act of judgment but, rather, brings into play a *cycle* of reflective, practical, and determining judgments.

3.4 Precisioning: Kant used a Latin word, *praecisionis*, to explain what the precisioning functional of approval of taste means. Literally, *praecisionis* means the act of lopping off or amputating an extremity. This is what a person does when he makes a concept precise. He lops off connections and contexts that are outside of the scope of the concept being made precise. It is this act Kant meant by "promoting the distinction of a manifold from other possible manifolds."

The precisioning functional *sets a concept in a distinct context by ascribing to this context a subjective necessity*. By doing so, it makes the object of a concept stand out as exemplary. An exemplary object is one that *serves as a pattern* in some context.

The precisioning functional is a Modality functional. This means that in judgments of taste precisioning pertains to *the relationship of the person to the concept* and not to how the concept represents its object. The other headings (Quantity, Quality, and Relation) pertain to the representing of an object. The

Modality heading pertains to the manner in which the person views the concepts of the object, e.g., as possible or impossible, actual or unreal, necessary or contingent.

While it is tempting to think of the process of sensibility as a process dealing with "one thing at a time," this is not correct. Sensibility deals with multiple concurrent representations, most of them unconscious (or, if you prefer, "preconscious" in the Freudian context [Freud (1915)]). If it were otherwise, the phenomenon of human creativity would be unexplainable. One hits closer to the mark by regarding sensibility as a process that is something like a maelstrom of representable matter. From this maelstrom imagination (in the process of apprehension) must pluck out perceptions. For describing this synthesis, Kant introduced the notion of what he called an *aesthetic Idea*:

> *Spirit* in an aesthetic significance is called the animating principle in the mind. That, however, by which this principle animates mind, the stuff which it uses for this, is that which suitably sets mental powers soaring, i.e., into a play that is self-maintaining and even strengthens the powers to that end.
>
> Now I maintain that this principle is nothing other than the capacity for presentation of *aesthetic Ideas*; by an aesthetic Idea, however, I mean that representation of imagination that occasions much thinking though without it being possible for any determinate thought, i.e., *concept*, to be adequate to it, which, consequently, no language fully attains or can make intelligible. . . .
>
> Now, if to a concept were imparted a representation of imagination that belongs to its presentation, but which by itself gives rise to so much thinking that it can never become concentrated in a determinate concept, hence which aesthetically enlarges the concept itself in an unbounded way, then here imagination is creative, and brings the power of intellectual ideas (reason) into motion, that is, at the instigation of a representation it gives more to think about than can be grasped and made distinct in it (although it does, to be sure, belong to the concept of the object). [Kant (1790) 5: 313-315]

Because aesthetic Ideas are representations that give rise to so much to think about that they cannot be contained in a single concept, precisioning is that in judgmentation which, by 'lopping off' part of the aesthetic Idea, allows intuitions to be apprehended and keeps the person from being overwhelmed by feelings of sublimity. The feeling of sublimity is an act of aesthetical reflective judgment that occurs when a person is unable to concentrate all that he is trying to apprehend in one intuition. Kant wrote,

> Nature is thus sublime in those of its appearances whose intuition brings with it the Idea of its infinity. Now the latter cannot happen except through the inadequacy of even the greatest effort of our power of imagination in the estimation of the magnitude of an object. . . . Thus it must be the *aesthetic* estimation of magnitude in which is felt the effort at concentration which exceeds the capacity of imagination to comprehend progressive apprehension in one whole of intuition [Kant (1790) 5: 255].

The feeling of sublimity is a feeling disturbing to equilibrium. The accommodation of perceptions in judgmentation attempts to equilibrate this disturbance. Being unable to concentrate the data of sensibility all in a single intuition, and thereby produce an intuitive representation expedient for equilibrium, the act of precisioning is the sole recourse left in judgmentation for settling disturbances raised by aesthetic Ideas. The act of precisioning can in this sense properly be called a psychologically necessary approval of a judgment of taste.

However, because precisioning is a Modality functional, *how* an intuition and its concept is *made* precise by the thinking Subject carries no guarantee that the concept formed will have real objective validity. This is vividly illustrated by various childish conceptions of causality Piaget documented [Piaget (1929); Piaget (1930)]. Precisioning combined with the satisficing character of human judgmentation lies at the root of all superstitions. But it also lies at the root of mathematics as a language for saying things *precisely* and in such a way that conclusions and deductions can be drawn from what is said.

§4. Summary

This chapter began with the questions "What is the basis for mathematics in the phenomenon of mind?" and "Why is mathematics the way it is?" The answers – perhaps surprisingly, perhaps not – are that subjective (aesthetic) judgments are the basis for all mathematics, and that mathematics is the way it is because the process of reflective judgments – particularly, those judgments called judgments of taste – are the only types of judgments that give rise to general concepts from particular ones. Human creativity is grounded in them and so mathematical science is also, and simultaneously, mathematical *art*.

Some might question, even recoil, at the idea that mathematics is grounded in subjective factors. But it could not be otherwise because all mathematical objects are supersensible. They are not given to us by means of our external senses; no, they are products of spontaneity in thinking – the reintroduction of concepts into the synthesis of imagination. The form of mathematical objects subsists in patterns; the matter subsists in syncretic coalescing of representations into representations of objects of imagination. These objects are placed in contexts by conceptualizing and *made* precise by precisioning. The word "rigor" denotes strict precision, and intuitions of mathematical objects are given contextual rigor and objective precision through these two characteristics of judgments of taste.

Forming a *general* (higher) concept always proceeds as a prosyllogism. It progresses from lower concepts (*examples*) *to* the more abstract (higher) concept. Examples are propaedeutic to not only the invention of new mathematical objects but also for *effective* teaching of mathematics. I think it likely that a great many more people would be skilled mathematicians if mathematics textbooks and teachers abandoned the approach called "formalism" and instead made more use of examples in ways appealing to the learner's affectivity [Wells (2014), chap. 14].

Empirical science differs from "the technical arts" and from craftsmanship in that empirical science is an art of *explaining things by means of theories*. Theories, however, are constructed using mathematical ideas and objects. Kant wrote,

> I assert, however, that in any special doctrine of nature there can be only as much *proper* science as there is *mathematics* therein. [Kant (1786) 4: 470]

STEM ("science, technology, engineering and math") is presently a popular idea and movement in today's American educational system. But the *practice* of science, technology innovation, engineering, and mathematics are creative practices that crucially depend *at root* on the aesthetic character of the human phenomenon of mind. For that reason, it is important to not diminish the emphasis of other frameworks of education, such as language arts and music, because these likewise cultivate aesthetical talents.

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